ISRAEL JOURNAL OF MATHEMATICS **230** (2019), 141–152
DOI: 10.1007/s11856-018-1811-0

SEPARABLE UNIVERSAL BANACH LATTICES

BY

DENNY H. LEUNG

Department of Mathematics, National University of Singapore
Singapore 119076
e-mail: matlhh@nus.edu.sg

AND

LEI LI

School of Mathematical Sciences and LPMC
Nankai University, Tianjin 300071, China
e-mail: leilee@nankai.edu.cn

AND

TIMUR OIKHBERG AND MARY ANGELICA TURSI

Department of Mathematics, University of Illinois
Urbana IL 61801, USA
e-mail: oikhberg@illinois.edu, gramcko2@illinois.edu

ABSTRACT

We construct separable universal injective and projective lattices for the class of all separable Banach lattices.

Received November 27, 2017 and in revised form April 18, 2018

1. Introduction

The object of this paper is to construct universal injective and projective objects for the class of separable (real) Banach lattices.

It is well known that $C[0,1]$ is a universal injective Banach space for the class of all separable Banach spaces—that is, any separable Banach space embeds isometrically into $C[0,1]$. Similarly, ℓ_1 is a universal projective Banach space for the class of separable Banach spaces—every separable Banach space is a quotient of ℓ_1. We construct similar objects in the lattice setting.

Below we briefly recall some essential notation. The reader is referred to [5] or [6] for more information about Banach lattices.

Suppose E and F are Banach lattices. We say that $u \in B(E,F)$ is a **lattice homomorphism** if it preserves lattice operations (it suffices to check that $u(x_1 \vee x_2) = ux_1 \vee ux_2$ for any $x_1, x_2 \in E$; note that u is necessarily positive). An operator which is both an isometry and a lattice homomorphism is referred to as a **lattice isometry**.

We call $q \in B(E,F)$ a **lattice quotient** if there is an ideal $I \subset E$ so that q identifies F with E/I. Notice that q is a lattice quotient if and only if it has the following properties: (i) q maps the open ball of E onto the open ball of F, and (ii) q is a lattice homomorphism. Indeed, in this case the formal identity $i : E/I \to F$ is a lattice isometry; by [1], the same is true for i^{-1}.

Throughout, we work with real lattices. We make use of two compact metrizable sets—the Hilbert cube $\mathbb{H}$, and the Cantor set Δ (that can be regarded as $[0,1]^{\mathbb{N}}$, respectively $\{0,1\}^{\mathbb{N}}$, equipped with the product topology). We use L_1 as a shorthand for $L_1(0,1)$.

The two theorems below represent the main results of this note.

Theorem 1.1: *The Banach lattice $C(\Delta, L_1)$ is injectively universal for the class of separable Banach lattices. That is, any separable Banach lattice embeds lattice isometrically into $C(\Delta, L_1)$.*

Theorem 1.2: *There exists a separable Banach lattice X which is projectively universal for the class of separable Banach lattices, that is, any separable Banach lattice is lattice isometric to a quotient of X by a closed lattice ideal.*

The proofs of Theorems 1.1 and 1.2 are given below.

Remark 1.1: As a separable Banach lattice can have infinitely many generators, no universal projective lattice can be finitely generated. However, the universal injective lattice $C(\Delta, L_1)$ can be generated by two elements. To verify this, we use a technique similar to [6, Theorem V.2.10]. Recall that L_1 is lattice isometric to $L_1(\Delta,\mu)$, where μ is the Haar measure on Δ (see [3, §14–15]). The measure μ can also be described as follows: consider $\nu = (\delta_0+\delta_1)/2$ (a probability measure on $\{0,1\}$); then $\mu = \nu^{\mathbb{N}}$ is a probability measure on $\Delta = \{0,1\}^{\mathbb{N}}$. Note that the set $K = \Delta\times\Delta$ is homeomorphic to Δ. Representing Δ as a compact subset of $\mathbb{R}$, and applying Stone's Theorem (see [6, Theorem II.7.3]), we observe that $C(\Delta)$ is generated by the identity $\mathbf{1}$ and the coordinate function. Thus,

$$C(K) \cong C(\Delta)$$

has two generators. To show that $C(K)$ is dense in $C(\Delta, L_1(\Delta,\mu))$, note that any $f \in C(\Delta, L_1(\Delta,\mu))$ is uniformly continuous. Hence, the functions of the form $\sum_{k=1}^n \chi_{A_k}\otimes f_k$ (where $f_k \in L_1(\Delta,\mu)$, and A_k is a clopen subset of Δ) are dense in $C(\Delta, L_1(\Delta,\mu))$.

2. The proof of Theorem 1.1

Let $A_n, n\in\mathbb{N}$, be finite nonempty sets and let $\widehat{T}$ be the tree $\bigcup_{k=0}^\infty \prod_{n=1}^k A_n$, where, as usual, the product $\prod_{n=1}^k A_n$ is defined to be $\emptyset$ if $k=0$. Suppose that $\sigma = (a_1,\dots,a_k)\in\prod_{n=1}^k A_n$; we say that σ has **length** k and write $|\sigma| = k$. For any $b\in A_{k+1}$, we denote the element $(a_1,\dots,a_k,b)\in\prod_{n=1}^{k+1} A_n$ by (σ,b).

Let E be a Banach lattice. A family $(x_\sigma)_{\sigma\in\widehat{T}}$ is said to be a **finitely branching tree** in E_+ if

(a) $x_\sigma\in E_+$ for all $\sigma\in\widehat{T}$,
(b) for any $\sigma\in\widehat{T}$ with $|\sigma| = k$, $(x_{(\sigma,b)})_{b\in A_{k+1}}$ is pairwise disjoint and

$$x_\sigma = \sum_{b\in A_{k+1}} x_{(\sigma,b)}.$$

Observe that if $(x_\sigma)_{\sigma\in\widehat{T}}$ is a finitely branching tree in E_+, then by (b), $\operatorname{span}\{x_\sigma : \sigma\in\widehat{T}\}$ is a vector sublattice of E.

Proposition 2.1: *Let E be a Banach lattice. Suppose that there is a finitely branching tree $(x_\sigma)_{\sigma\in\widehat{T}}$ in E_+ so that E is the closed linear span of $(x_\sigma)_{\sigma\in\widehat{T}}$. Then there exists a compact metric space K so that E is a lattice isometric to a closed sublattice of $C(K, L_1)$.*

Proof. Obviously, under the given assumption, E is a separable Banach lattice. Let K be the positive part of the closed ball of E^*, endowed with the weak* topology. Then K is a compact metrizable topological space. By rescaling if necessary, we may assume that $\|x_\emptyset\| \leq 1$. For each $\sigma \in \widehat{T}$, the function $g_\sigma : K \to \mathbb{R}$ given by $g_\sigma(x^*) = x^*(x_\sigma)$ is a nonnegative continuous function on K. Furthermore, for all $\sigma \in \widehat{T}$ with $|\sigma| = k$, it follows from property (b) that

$$g_\sigma = \sum_{b \in A_{k+1}} g_{(\sigma,b)}. \tag{1}$$

We now define functions $f_\sigma : K \to L_1$, $\sigma \in \widehat{T}$, inductively as follows. Let $f_\emptyset(x^*) = \chi_{[0,g_\emptyset(x^*)]}$. By the continuity of $g_\emptyset$, we see that $f_\emptyset$ is a continuous function from K into L_1. In general, assume that f_σ has been defined so that $f_\sigma(x^*) = \chi_{[c(x^*),d(x^*)]}$, where $c, d : K \to \mathbb{R}$ are nonnegative continuous functions so that $d - c = g_\sigma$. Label the elements in A_{k+1} as $b_1, \dots, b_r$. Define $f_{(\sigma,b_i)}(x^*)$, $1 \leq i \leq r$, to be the characteristic function of the interval

$$\Big[c(x^*) + \sum_{j=1}^{i-1} g_{(\sigma,b_j)}(x^*),\, c(x^*) + \sum_{j=1}^{i} g_{(\sigma,b_j)}(x^*)\Big].$$

By continuity of c and $g_{(\sigma,b_j)}$, $f_{(\sigma,b_i)}$ is a continuous function from K into L_1 for each i. This completes the inductive definition of f_σ, $\sigma \in \widehat{T}$. It follows from (1) that

$$f_\sigma = \sum_{b \in A_{k+1}} f_{(\sigma,b)} \quad \text{if } |\sigma| = k \tag{2}$$

(equality in the L_1 sense at each $x^* \in K$). From (b) and (2), we see that the map $x_\sigma \mapsto f_\sigma$, $\sigma \in \widehat{T}$, extends to a linear map u from $\operatorname{span}\{x_\sigma : \sigma \in \widehat{T}\}$ into $C(K, L_1)$. By (b), for any $y \in \operatorname{span}\{x_\sigma : \sigma \in \widehat{T}\}$, one can derive that $y \in \operatorname{span}\{x_\sigma : |\sigma| = k\}$ for all sufficiently large k. In particular, $\operatorname{span}\{x_\sigma : \sigma \in \widehat{T}\}$ is a sublattice of E. Also, it is easy to check that if σ and τ are distinct elements in $\widehat{T}$ of the same length, then $f_\sigma(x^*) \wedge f_\tau(x^*) = 0$ (in L_1) for each $x^* \in K$. It follows that the map u is a lattice homomorphism. Next, we show that u is an (into) isometry. Let $x \in \operatorname{span}\{x_\sigma : \sigma \in \widehat{T}\}$. Write $x = \sum_{|\sigma|=k} c_\sigma x_\sigma$ for some $k \in \mathbb{N}$ and $c_\sigma \in \mathbb{R}$. Then $|x| = \sum_{|\sigma|=k} |c_\sigma| x_\sigma$ and

$$|ux| = u|x| = \sum_{|\sigma|=k} |c_\sigma| f_\sigma.$$

By construction, $\|f_\sigma(x^*)\|_{L_1} = g_\sigma(x^*) = x^*(x_\sigma)$. Since K is the positive part of the ball of E^*, one can derive that

$$\begin{aligned}\|ux\| =& |||ux||| = \sup_{x^*\in K}\Big\| \sum_{|\sigma|=k} |c_\sigma| f_\sigma(x^*)\Big\|_{L_1}\\ &= \sup_{x^*\in K} \sum_{|\sigma|=k} |c_\sigma| \|f_\sigma(x^*)\|_{L_1}\\ &= \sup_{x^*\in K} \sum_{|\sigma|=k} |c_\sigma| x^*(x_\sigma) = \sup_{x^*\in K} x^*\Big(\sum_{|\sigma|=k} |c_\sigma| x_\sigma\Big)\\ &= \sup_{x^*\in K} x^*(|x|) = |||x|||\\ &=\|x\|.\end{aligned}$$

Hence u is a lattice isometry from $\operatorname{span}\{x_\sigma : \sigma \in \widehat{T}\}$ into $C(K, L_1)$. As $\operatorname{span}\{x_\sigma : \sigma \in \widehat{T}\}$ is dense in E by assumption, u extends to a lattice isometry from E into $C(K, L_1)$. ∎

Proposition 2.2: *Let E be a separable Banach lattice, regarded as a closed sublattice of its bidual E^{**}. There is a Banach lattice F such that $E \subseteq F \subseteq E^{**}$, F_+ contains a finitely branching tree $(x_\sigma)_{\sigma\in\widehat{T}}$ and* $\operatorname{span}\{x_\sigma : \sigma \in \widehat{T}\}$ *is dense in F.*

Proof. Let $(e_i)_{i=1}^\infty$ be a countable dense subset of E consisting of nonzero vectors. We shall construct recursively a finitely branching tree $(x_\sigma)_{\sigma\in\widehat{T}} \subset E_+^{**}$ so that, for any $1 \le m \le n$,

$$\operatorname{dist}(e_m, \operatorname{span}\{x_\sigma : |\sigma| = n\}) < 2^{-n}.$$

Then the proposition follows by taking F to be the closed linear span of $(x_\sigma)_{\sigma\in\widehat{T}}$ in E^{**}.

Start the construction by setting $A_0 = \emptyset$ and

$$x_\emptyset = e = \sum_{i=1}^\infty \frac{|e_i|}{2^i\|e_i\|}.$$

Suppose that $n \in \mathbb{N} \cup \{0\}$ and the sets $A_0, A_1, \ldots, A_n$ and vectors $x_\sigma \in E_+^{**}$ ($|\sigma| \le n$) have already been selected so that condition (b) above is satisfied for every σ with $|\sigma| < n$. In particular,

$$\sum_{|\sigma|=n} x_\sigma = e.$$

Since for all $1 \leq i \leq n+1$, e_i lies in the principal ideal generated by e in E^{**}, by Freudenthal's Spectral Theorem [5, Theorem 1.2.18] and its proof, there exist mutually disjoint $z_1, \ldots, z_N \in E^{**}_+$ so that $z_1 + \cdots + z_N = e$, and

$$\operatorname{dist}(e_m, \operatorname{span}\{z_1, \ldots, z_N\}) < 2^{-(n+1)}$$

for $1 \leq m \leq n+1$. Denote by P_i the band projection from E^{**} onto the band generated by z_i in E^{**}, $1 \leq i \leq N$. Let $A_{n+1} = \{1, \ldots, N\}$; for $\sigma \in \prod_{k=1}^n A_k$ and $i \in A_{n+1}$, let $x_{(\sigma,i)} = P_i x_\sigma$. Since x_σ lies in the band B generated by e in E^{**} and $\sum_{i=1}^N P_i$ is the band projection onto B, $x_\sigma = \sum_{i \in A_{n+1}} x_{(\sigma,i)}$. This completes the inductive construction of $(x_\sigma)_{\sigma \in \widehat{T}}$, where $\widehat{T} = \bigcup_{k=0}^\infty \prod_{n=1}^k A_n$. Clearly, $(x_\sigma)_{\sigma \in \widehat{T}}$ is a finitely branching tree in E_+. Furthermore, in the notation above,

$$z_i = P_i e = \sum_{|\sigma|=n} P_i x_\sigma = \sum_{|\sigma|=n} x_{(\sigma,i)}.$$

Thus, for $1 \leq m \leq n+1$,

$$\begin{aligned}\operatorname{dist}(e_m, \operatorname{span}\{x_\sigma : |\sigma| = n+1\}) &\leq \operatorname{dist}(e_m, \operatorname{span}\{z_1, \ldots, z_N\}) \\ &< 2^{-(n+1)}. \qquad \blacksquare\end{aligned}$$

Proof of Theorem 1.1. By Propositions 2.1 and 2.2, there are a compact metric space K and a lattice isometry u from E into $C(K, L_1)$. It is well known that there exists a continuous surjection $\pi : \Delta \to K$. Then the map $j : E \to C(\Delta, L_1)$ given by $jx = ux \circ \pi$ is a lattice isometry. $\blacksquare$

3. The proof of Theorem 1.2

A few words of motivation before we begin the proof proper. Suppose that X is a separable Banach lattice that is projectively universal for the class of separable Banach lattices. For any separable Banach lattice E, there is a lattice quotient q from X onto E. Then $q^* B_{E^*}$ is a $\sigma(X^*, X)$-closed convex solid subset of the $\sigma(X^*, X)$-compact metrizable space B_{X^*}. Let $\mathbb{H}$ be the Hilbert cube $[0,1]^{\mathbb{N}}$. For each separable Banach lattice E, we will present B_{E^*} as a closed convex solid subset of the ball of $M(\mathbb{H}) = C(\mathbb{H})^*$ on a different copy of $\mathbb{H}$. We then stitch these copies together to form a compact metric space, say K. The space X is taken to be the completion of $C(K)$ normed by the union of the copies of B_{E^*}.

If V is a solid subset of $B_{M(\mathbb{H})}$, define a seminorm ρ_V on $C(\mathbb{H})$ by

$$\rho_V(f) = \sup_{\mu \in V} \left| \int f \, d\mu \right|.$$

Since V is solid, ρ_V is a lattice seminorm and $\ker \rho_V$ is a vector lattice ideal of $C(\mathbb{H})$. Thus $C(\mathbb{H})/\ker \rho_V$ is a vector lattice. Clearly, ρ_V induces a lattice norm on $C(\mathbb{H})/\ker \rho_V$, which we denote by $\widetilde{\rho}_V$.

Proposition 3.1: *Let E be a separable Banach lattice. Then there exists a $\sigma(M(\mathbb{H}), C(\mathbb{H}))$-closed convex solid subset V_E of $B_{M(\mathbb{H})}$ such that E is lattice isometric to the completion of $C(\mathbb{H})/\ker \rho_{V_E}$ with respect to the lattice norm $\widetilde{\rho}_{V_E}$.*

Proof. Choose a sequence (x_n) in B_{E^+} that is dense in B_{E^+}. Set $x = \sum \frac{x_n}{2^n}$. There are a compact Hausdorff space L and a vector lattice isomorphism i from $C(L)$ onto the ideal $E_x = \bigcup_k [-kx, kx]$ of E. Furthermore, $x = i1_L$, where 1_L is the constant function with value 1. Since $x_n \in E_x$, $x_n = if_n$ for some $f_n \in C(L)$. Let F be the closed (with respect to the sup-norm) sublattice of $C(L)$ generated by $(f_n) \cup \{1_L\}$. Since F is an AM-space with unit, there are a compact Hausdorff space K and a Banach lattice isomorphism j from $C(K)$ onto F such that $j1_K = 1_L$. The closed sublattice generated by a countable set is separable [4]; see also [6, p. 143, Exercise 5(c)]. Hence F is separable and thus K is metrizable. By [2, Theorem 4.14], there is an (into) homeomorphism $\varphi : K \to \mathbb{H}$. Define $q : C(\mathbb{H}) \to C(K)$ by $qf = f \circ \varphi$. Then $T = i \circ j \circ q : C(\mathbb{H}) \to E$ is a vector lattice homomorphism and, in particular, a bounded linear operator. Furthermore, $TB_{C(\mathbb{H})} \subseteq [-x, x]$ and $\|x\| \leq 1$. Thus $\|T\| \leq 1$. Set $V_E = T^* B_{E^*}$. Then V_E is a $\sigma(M(\mathbb{H}), C(\mathbb{H}))$-closed convex subset of $B_{M(\mathbb{H})}$.

Next, we show that V_E is solid in $M(\mathbb{H})$. Suppose that $|\nu| \leq |\mu|$, where $\nu \in M(\mathbb{H})$ and $\mu \in V_E$. Choose $x^* \in B_{E^*}$ so that $\mu = T^* x^*$. For $f \in C(\mathbb{H})$, if $|g| \leq |f|$ we have that $|Tg| = T|g| \leq T|f|$ which implies that

$$\begin{aligned} |\langle f, \nu \rangle| &\leq \langle |f|, |\nu| \rangle \leq \langle |f|, |\mu| \rangle \\ &= \sup_{|g| \leq |f|} |\langle g, \mu \rangle| = \sup_{|g| \leq |f|} |\langle Tg, x^* \rangle| \\ &\leq \langle T|f|, |x^*| \rangle \\ &\leq \|T|f|\| \, \|x^*\| = \|Tf\| \|x^*\|. \end{aligned}$$

It follows that $y^* : T(C(\mathbb{H})) \to \mathbb{R}$ given by $y^*(Tf) = \langle f, \nu \rangle$ defines a bounded linear functional on the subspace $T(C(\mathbb{H}))$ of E. Since $x_n \in T(C(\mathbb{H}))$ for all n, $T(C(\mathbb{H}))$ is a dense subspace of E. Thus y^* extends uniquely to an element in E^*, which we denote still by y^*. By the computation above, $\|y^*\| \leq \|x^*\|$ and hence $y^* \in B_{E^*}$. Clearly, it follows from the definition that $T^* y^* = \nu$. Hence $\nu \in V_E$, as desired.

Finally, we show that the map $S : (C(\mathbb{H})/\ker \rho_{V_E}, \widetilde{\rho}_{V_E}) \to E$ given by

$$S\widetilde{f} = Tf$$

is a well-defined into lattice isometry. Since the image of S is $T(C(\mathbb{H}))$ and hence dense in E, the proof would be complete. If $f \in \ker \rho_{V_E}$, then $\langle f, T^* x^* \rangle = 0$ for all $x^* \in B_{E^*}$. Thus $Tf = 0$. This shows that S is well-defined. Furthermore, for any $\widetilde{f} \in C(\mathbb{H})/\ker \rho_{V_E}$,

$$\widetilde{\rho}_{V_E}(\widetilde{f}) = \rho_{V_E}(f) = \sup_{x^* \in B_{E^*}} |\langle f, T^* x^* \rangle| = \|Tf\| = \|S\widetilde{f}\|.$$

Hence S is an into isometry. Also,

$$|S\widetilde{f}| = |Tf| = T|f| = S|\widetilde{f}|.$$

Therefore, S is a lattice homomorphism. ∎

Since $C(\mathbb{H})$ is separable, we have that $B_{M(\mathbb{H})}$ is a compact metric space in the $\sigma(M(\mathbb{H}), C(\mathbb{H}))$-topology. Let d be a metric on $B_{M(\mathbb{H})}$ that gives the $\sigma(M(\mathbb{H}), C(\mathbb{H}))$-topology. By a theorem of Hausdorff (see [2, Theorem 4.26]), the set $\mathcal{C}$ of all $\sigma(M(\mathbb{H}), C(\mathbb{H}))$-closed subsets of $B_{M(\mathbb{H})}$ is compact with respect to the Hausdorff metric D generated by d. Let $f \in C(\mathbb{H})$. Then there is a metric d' on $B_{M(\mathbb{H})}$ that gives the $\sigma(M(\mathbb{H}), C(\mathbb{H}))$-topology and that

$$d'(\mu, \nu) \geq |\langle f, \mu \rangle - \langle f, \nu \rangle| \quad \text{for all } \mu, \nu \in B_{M(\mathbb{H})}.$$

Since $B_{M(\mathbb{H})}$ is $\sigma(M(\mathbb{H}), C(\mathbb{H}))$-compact, the formal identity map from $(B_{M(\mathbb{H})}, d)$ to $(B_{M(\mathbb{H})}, d')$ is a uniform homeomorphism. Thus, if D' is the Hausdorff metric on $\mathcal{C}$ generated by d', then D and D' yield the same topology on $\mathcal{C}$.

Proposition 3.2: *Let $\mathcal{K}$ be the set of all $\sigma(M(\mathbb{H}), C(\mathbb{H}))$-closed convex solid subsets of $B_{M(\mathbb{H})}$. Then $\mathcal{K}$ is a closed subset of $\mathcal{C}$. Consequently, $\mathcal{K}$ is a compact set with respect to the Hausdorff metric D generated by d.*

Proof. Suppose that $V_n \in \mathcal{K}$ for all n and that $D(V_n, V) \to 0$ for some $V \in \mathcal{C}$. It is easy to see that V is convex. Indeed, suppose that $a, b \in V$ and $0 \le \alpha \le 1$. There are sequences (v_n) and (w_n) so that $v_n, w_n \in V_n$ for each $n \in \mathbb{N}$ and that $d(v_n, a), d(w_n, b) \to 0$, i.e., $v_n \to a$ and $w_n \to b$ with respect to $\sigma(M(\mathbb{H}), C(\mathbb{H}))$. Then

$$\alpha v_n + (1-\alpha) w_n \to \alpha a + (1-\alpha) b \quad \text{with respect to } \sigma(M(\mathbb{H}), C(\mathbb{H})).$$

Since each V_n is convex, $\alpha v_n + (1-\alpha) w_n \in V_n$. Hence

$$d(\alpha v_n + (1-\alpha) w_n, V) \le D(V_n, V) \to 0.$$

Choose $u_n \in V$ such that $d(\alpha v_n + (1-\alpha) w_n, u_n) \to 0$. Then $u_n \to \alpha a + (1-\alpha) b$ with respect to $\sigma(M(\mathbb{H}), C(\mathbb{H}))$. Hence $\alpha a + (1-\alpha) b \in V$. Similarly, one can show that V is symmetric.

Next, we show that V is solid (in $B_{M(\mathbb{H})}$). Suppose on the contrary that there are a, b so that $|a| \le |b|$, $b \in V$ and $a \notin V$. Since V is convex, symmetric and $\sigma(M(\mathbb{H}), C(\mathbb{H}))$-closed, there exists $f \in C(\mathbb{H})$ so that

$$\langle f, a \rangle > \sup_{v \in V} |\langle f, v \rangle|.$$

As discussed above, there is a metric d' on $B_{M(\mathbb{H})}$ so that its Hausdorff metric D' generates the same topology on $\mathcal{C}$ and that

$$d'(v_1, v_2) \ge |\langle f, v_1 \rangle - \langle f, v_2 \rangle| \quad \text{for all } v_1, v_2 \in B_{M(\mathbb{H})}.$$

Let $w \in V_n$. Since V_n is solid,

$$\begin{aligned} \langle |f|, |w| \rangle &= \sup_{|u| \le |w|} |\langle f, u \rangle| \\ &\le \sup_{u \in V_n} |\langle f, u \rangle| \\ &\le \sup_{v \in V} |\langle f, v \rangle| + D'(V_n, V). \end{aligned}$$

Choose (x_n) so that $x_n \in V_n$ for each n and that $d'(x_n, b) \to 0$. For any $\varepsilon > 0$, there exists g with $|g| \le |f|$ such that

$$|\langle g, b \rangle| + \varepsilon \ge \langle |f|, |b| \rangle.$$

We have

$$|\langle g, b \rangle| = \lim |\langle g, x_n \rangle| \le \limsup \langle |f|, |x_n| \rangle.$$

It follows that

$$\begin{aligned}\langle f, a\rangle &\leq \langle |f|, |a|\rangle \leq \langle |f|, |b|\rangle \\ &\leq \limsup_n \langle |f|, |x_n|\rangle \\ &\leq \limsup_n (\sup_{v\in V} |\langle f, v\rangle| + D'(V_n, V)) \\ &= \sup_{v\in V} |\langle f, v\rangle|,\end{aligned}$$

contrary to the choice of f. This proves that V is solid. ∎

Fix $V \in \mathcal{K}$. Define $q_V : C(\mathcal{K} \times \mathbb{H}) \to C(\mathbb{H})$ by $q_V(f) = f_{|\{V\}\times\mathbb{H}}$. Let $\mathcal{B}$ be the set $\bigcup_{V\in\mathcal{K}} q_V^*(V)$ and define $\rho_{\mathcal{B}} : C(\mathcal{K} \times \mathbb{H}) \to \mathbb{R}$ by

$$\rho_{\mathcal{B}}(F) = \sup_{\mu\in\mathcal{B}} |\int F\, d\mu|.$$

Lemma 3.3: *$\rho_{\mathcal{B}}$ is a lattice seminorm on $C(\mathcal{K} \times \mathbb{H})$. Thus $C(\mathcal{K} \times \mathbb{H})/\ker\rho_{\mathcal{B}}$ is a vector lattice. Denote the lattice norm induced by $\rho_{\mathcal{B}}$ on $C(\mathcal{K} \times \mathbb{H})/\ker\rho_{\mathcal{B}}$ by $\widetilde{\rho}_{\mathcal{B}}$. The completion X of $C(\mathcal{K}\times\mathbb{H})/\ker\rho_{\mathcal{B}}$ with respect to $\widetilde{\rho}_{\mathcal{B}}$ is a separable Banach lattice.*

Proof. Since $\mathcal{K} \times \mathbb{H}$ is a compact metric space, $C(\mathcal{K} \times \mathbb{H})$ is separable with respect to the sup-norm. If $V \in \mathcal{K}$, then $V \subseteq B_{M(\mathbb{H})}$ and it is clear that $q_V^*(V) \subseteq B_{M(\mathcal{K}\times\mathbb{H})}$. Hence $\mathcal{B} \subseteq B_{M(\mathcal{K}\times\mathbb{H})}$. It is then clear that $\rho_{\mathcal{B}} \leq \|\cdot\|_\infty$. Let A be a countable dense subset of $C(\mathcal{K} \times \mathbb{H})$ with respect to the sup-norm. Then $\{\widetilde{F} : F \in A\}$ is a countable dense subset of $C(\mathcal{K}\times\mathbb{H})/\ker\rho_{\mathcal{B}}$ with respect to $\widetilde{\rho}_{\mathcal{B}}$. Thus X is separable. ∎

If $V \in \mathcal{K}$, identify $\{V\} \times \mathbb{H}$ with $\mathbb{H}$.

Lemma 3.4: *Let E be a separable Banach lattice. The map $Q : C(\mathcal{K}\times\mathbb{H}) \to C(\mathbb{H})$ given by*

$$QF = F_{|\{V_E\}\times\mathbb{H}}$$

has the following properties:

(1) *$Q(\ker\rho_{\mathcal{B}}) \subseteq \ker\rho_{V_E}$ and hence Q induces a map*

$$\widetilde{Q} : C(\mathcal{K} \times \mathbb{H})/\ker\rho_{\mathcal{B}} \to C(\mathbb{H})/\ker\rho_{V_E}.$$

$\widetilde{Q}$ is a lattice homomorphism.

(2) *$\widetilde{Q}$ maps the open ball in $(C(\mathcal{K} \times \mathbb{H})/\ker\rho_{\mathcal{B}}, \widetilde{\rho}_{\mathcal{B}})$ onto the open ball in $(C(\mathbb{H})/\ker\rho_{V_E}, \widetilde{\rho}_{V_E})$.*

Proof. (1) Let $F \in \ker \rho_{\mathcal{B}}$. Thus $\int F\,d\mu = 0$ for all $\mu \in \mathcal{B}$. In particular, $\int F\,d\mu = 0$ for all $\mu \in q^*_{V_E}(V_E)$. Let $f = QF = F_{|\{V_E\}\times\mathbb{H}}$ and identify $\{V_E\}\times\mathbb{H}$ with $\mathbb{H}$. If $\nu \in V_E$, let $\mu = q^*_{V_E}(\nu)$. We have

$$0 = \int F\,d\mu = \int q_{V_E}F\,d\nu = \int f\,d\nu.$$

This shows that $\rho_{V_E}(QF) = 0$. Since Q is obviously a lattice homomorphism, so is $\widetilde{Q}$.

(2) Let $\widetilde{F} \in C(\mathcal{K}\times\mathbb{H})/\ker\rho_{\mathcal{B}}$ with $\widetilde{\rho}_{\mathcal{B}}(\widetilde{F}) < 1$. Then $F \in C(\mathcal{K}\times\mathbb{H})$ and $\rho_{\mathcal{B}}(F) < 1$. Let $f = F_{|\{V_E\}\times\mathbb{H}}$, identified as a function on $\mathbb{H}$. For any $\nu \in V_E$, $q^*_{V_E}(\nu) \in \mathcal{B}$ and hence

$$\left|\int f\,d\nu\right| = \left|\int q_{V_E}F\,d\nu\right| \le \rho_{\mathcal{B}}(F) < 1.$$

Thus

$$\rho_{V_E}(f) = \sup_{\nu\in V_E}\left|\int f\,d\nu\right| < 1.$$

We claim that the function $V \in \mathcal{K} \mapsto \rho_V(f) \in \mathbb{R}$ is continuous. As per the discussion preceding Proposition 3.2, there is a metric d' on $B_{M(\mathbb{H})}$ so that

$$d'(\nu_1,\nu_2) \ge \left|\int f\,d\nu_1 - \int f\,d\nu_2\right| \quad \text{for all } \nu_1,\nu_2 \in B_{M(\mathbb{H})}$$

and that the associated Hausdorff metric D' generates the same topology as D on $\mathcal{K}$. Suppose that $V, W \in \mathcal{K}$ and $D'(V,W) < \varepsilon$. Let $\nu \in V$. There exists $\nu' \in W$ such that

$$\left|\int f\,d\nu - \int f\,d\nu'\right| \le d'(\nu,\nu') < \varepsilon.$$

It follows that $\rho_V(f) \le \rho_W(f) + \varepsilon$. The claim follows by symmetry.

By continuity, there is an open neighborhood $\mathcal{O}$ of V_E in $\mathcal{K}$ such that

$$\sup_{V\in\mathcal{O}} \rho_V(f) < 1.$$

Choose a continuous function $h : \mathcal{K} \to [0,1]$ such that $h(V_E) = 1$ and that $h(V) = 0$ for all $V \notin \mathcal{O}$. Let G be the function on $\mathcal{K}\times\mathbb{H}$ defined by

$$G(V,x) = h(V)f(x).$$

Then $G \in C(\mathcal{K}\times\mathbb{H})$. We have

$$\rho_{\mathcal{B}}(G) = \sup_{V\in\mathcal{K}}\sup_{\nu\in V}\left|\int q_V(G)\,d\nu\right| = \sup_{V\in\mathcal{K}} h(V)\rho_V(f).$$

If $V \notin \mathcal{O}$, then $h(V) = 0$. Otherwise, $0 \le h(V) \le 1$. Hence

$$\rho_{\mathcal{B}}(G) \le \sup_{V \in \mathcal{O}} \rho_V(f) < 1.$$

This proves that $\widetilde{G}$ belongs to the open ball of $(C(\mathcal{K} \times \mathbb{H})/\ker \rho_{\mathcal{B}}, \widetilde{\rho}_{\mathcal{B}})$. Finally,

$$\widetilde{Q}\widetilde{G} = \widetilde{QG} = (G_{|\{V_E\}\times\mathbb{H}})^{\sim} = (h(V_E)f)^{\sim} = \widetilde{f} = \widetilde{F}. \qquad \blacksquare$$

Proof of Theorem 1.2. Let X be the separable Banach lattice defined in Lemma 3.3. Let E be a separable Banach lattice. By Proposition 3.1, there exists $V_E \in \mathcal{K}$ such that E is lattice isometric to the completion of $(C(\mathbb{H})/\ker \rho_{V_E}, \widetilde{\rho}_{V_E})$. We will identify E with the completion.

Define $\widetilde{Q}$ as in Lemma 3.4. By the lemma, $\widetilde{Q}$ extends uniquely to a lattice homomorphism $\mathbf{Q}$ that maps the open ball of X onto the open ball of E. Hence $\mathbf{Q}$ is a lattice quotient from X onto E. (See the Introduction.) $\blacksquare$

Acknowledgements. The authors wish to thank the organizers of Positivity IX, where part of this work was carried out. The first author is partially supported by AcRF Grant R-146-000-242-114. The second author is partially supported by NSF of China (11671214).

References

[1] Y. Abramovich, *Isometries of normed lattices*, Optimizatsiya **43** (1988), 74–80.

[2] A. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics, Vol. 156, Springer, New York, 1995.

[3] E. Lacey, *The Isometric Theory of Classical Banach Spaces*, Die Grundlehren der mathematischen Wissenschaften, Vol. 208, Springer, New York–Heidelberg, 1974.

[4] P. Meyer-Nieberg, *Zur Schwachen Kompaktheit in Banachverbänden*, Mathematische Zeitschrift **134** (1973), 303–315.

[5] P. Meyer-Nieberg, *Banach Lattices*, Universitext, Springer, Berlin, 1991.

[6] H. H. Schaefer, *Banach Lattices and Positive Operators*, Die Grundlehren der mathematischen Wissenschaften, Vol. 215, Springer, New York–Heidelberg, 1974.