APPROXIMATE ULTRAHOMOGENEITY IN L_pL_q LATTICES

MARY ANGELICA TURSI

Abstract. We show that for $1 \leq p, q < \infty$ with $p/q \notin \mathbb{N}$, the doubly atomless separable L_pL_q Banach lattice $L_p(L_q)$ is approximately ultrahomogeneous (AUH) over the class of its finitely generated sublattices. The above is not true when $p/q \in \mathbb{N}$ and $p \neq q$. However, for any $p \neq q$, $L_p(L_q)$ is AUH over the finitely generated lattices in the class BL_pL_q of bands of L_pL_q lattices.

1. Introduction

In this paper, we explore the homogeneity properties (or lack thereof) of the class of L_pL_q lattices under various conditions.

The following is taken from [6]: A Banach lattice X is an **abstract** L_pL_q **lattice** if there is a measure space (Ω, Σ, μ) such that X can be equipped with an $L_\infty(\Omega)$-module and a map $N : X \to L_p(\Omega)_+$ such that

- For all $\phi \in L_\infty(\Omega)_+$ and $x \in X_+$, $\phi \cdot x \geq 0$,
- For all $\phi \in L_\infty(\Omega)$ and $x \in X$, $N[\phi \cdot x] = |\phi| N[x]$.
- For all $x, y \in X$, $N[x + y] \leq N[x] + N[y]$
- If x and y are disjoint, $N[x + y]^q = N[x]^q + N[y]^q$, and if $|x| \leq |y|$, then $N[x] \leq N[y]$.
- For all $x \in X$, $\|x\| = \|N[x]\|_{L_p}$.

When the abstract L_pL_q space is separable, it has a concrete representation: Suppose (Ω, Σ, μ) and (Ω', Σ', μ') are measure spaces. Denote by $L_p(\Omega; L_q(\Omega'))$ the space of Bochner-measurable functions $f : \Omega \to L_q(\Omega')$ such that the function $N[f]$, with $N[f](\omega) = \|f(\omega)\|_q$ for $\omega \in \Omega$, is in $L_p(\Omega)$. The class of *bands* in L_pL_q lattices, which we denote by BL_pL_q, has certain analogous properties to those of L_p spaces, particularly with respect to its isometric theory.

L_pL_q lattices (and their sublattices) have been extensively studied for their model theoretic properties in [6] and [7]. It turns out that while abstract L_pL_q lattices themselves are not axiomatizable, the larger class BL_pL_q is axiomatizable with certain properties corresponding to those of L_p spaces. For instance, it is known that the class of atomless L_p lattices is separably categorical, meaning that there exists one unique atomless separable L_p lattice up to lattice isometry. Correspondingly, the class of *doubly atomless*

BL_pL_q lattices is also separably categorical; in particular, up to lattice isometry, $L_p([0,1];L_q[0,1])$, which throughout will just be referred to as $L_p(L_q)$, is the unique separable doubly atomless BL_pL_q lattice (see [7, Proposition 2.6]).

Additionally, when $p \neq q$, the lattice isometries of L_pL_q lattices can be characterized in a manner echoing those of linear isometries over L_p spaces (with $p \neq 2$). Recall from [1, Ch. 11 Theorem 5.1] that a map $T : L_p(0,1) \to L_p(0,1)$ is a surjective linear isometry iff $Tf(t) = h(t)f(\phi(t))$, where ϕ is a measure-preserving transformation and h is related to ϕ through Radon-Nikodym derivatives. If we want T to be a *lattice* isometry as well, then we also have h positive (and the above characterization will also work for $p = 2$). In [3] (for the case of $q = 2$) and [13], a corresponding characterization of linear isometries is found for spaces of the form $L_p(X;Y)$, for certain p and Banach spaces Y. In particular, for L_pL_q lattices with $p \neq q$: given $f \in L_p(\Omega; L_q(\Omega'))$, where f is understood as a map from Ω to L_q, any surjective linear isometry T is of the form

$$Tf(x) = S(x)\big(e(x)\phi f(x)\big),$$

where ϕ is a set isomorphism (see [3] and [13] for definitions) e is a measurable function related to ϕ via Radon-Nikodym derivatives, and S is a Bochner-measurable function from Ω to the space of linear maps from L_q to itself such that for each x, $S(x)$ is a linear isometry over L_q.

In [11], Raynaud obtained results on linear subspaces of L_pL_q spaces, showing that for $1 \leq q \leq p < \infty$, some ℓ_r linearly isomorphically embeds into $L_p(L_q)$ iff it embeds either to L_p or to L_q. However, when $1 \leq p \leq q < \infty$, for $p \leq r \leq q$, the space ℓ_r isometrically embeds as a lattice in $L_p(L_q)$, and for any p-convex and q-concave Orlicz function ϕ, the lattice L_ϕ embeds lattice isomorphically into $L_p(L_q)$. Thus, unlike with L_p lattices whose infinite dimensional sublattices are determined up to lattice isometry by the number of atoms, the sublattices of L_pL_q are not so simply classifiable.

In fact, the lattice isometry classes behave more like the L_p linear isometries, at least along the positive cone, as is evident in certain equimeasurability results for L_pL_q lattices. In [11], Raynaud also obtained the following on uniqueness of measures, a variation of a result which will be relevant in this paper: let $\alpha > 0, \alpha \notin \mathbb{N}$, and suppose two probability measures ν_1 and ν_2 on $\mathbb{R}_+$ are such that for all $s > 0$,

$$\int_0^\infty (t+s)^\alpha \, d\nu_1(t) = \int_0^\infty (t+s)^\alpha \, d\nu_1(t).$$

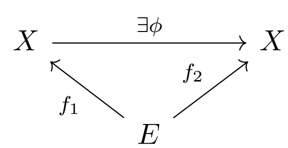
Then $\nu_1 = \nu_2$. Linde gives an alternate proof of this result in [8].

Various versions and expansions of the above result appear in reference to L_p spaces: for instance, an early result from Rudin generalizes the above to equality of integrals over $\mathbb{R}^n$: ([12]). Assume that $\alpha > 0$ with $\alpha \notin 2\mathbb{N}$, and suppose that for all $\mathbf{v} \in R^n$,

$$\int_{\mathbb{R}^n} (1 + \mathbf{v} \cdot z)^\alpha \, d\nu_1(z) = \int_{\mathbb{R}^n} (1 + \mathbf{v} \cdot z)^\alpha \, d\nu_2(z)$$

Then $\nu_1 = \nu_2$. An application of this result is a similar condition by which one can show that one collection of measurable functions $F : \mathbb{R}^n \to \mathbb{R}$, with $\mathbf{f} = (f_1, ..., f_n)$ is equimeasurable with another collection $\mathbf{g} = (g_1, ..., g_n)$ By defining ν_1 and ν_2 as pushforward measures of F and G. In the case of L_p spaces, if f and g are corresponding basic sequences whose pushforward measures satisfy the above for $\alpha = p$, then they generate isometric Banach spaces. Raynaud's result shows the converse is true for $\alpha \neq 4, 6, 8, ...$. A similar result in$L_p(L_q)$ from [7] holds for $\alpha = p/q \notin \mathbb{N}$ under certain conditions, except instead of equimeasurable $\mathbf{f}$ and $\mathbf{g}$, when the f_i's and $g_i's$ are mutually disjoint and positive and the map $f_i \mapsto g_i$ generates a lattice isometry, $(N[f_1], ..., N[f_n])$ and $(N[g_1], ..., N[g_n])$ are equimeasurable.

Recall that a space X is *approximately ultrahomogeneous* (AUH) over a class $\mathcal{G}$ of finitely generated spaces if for all appropriate embeddings $f_i; E \hookrightarrow X$ with $i = 1, 2$, for all $E \in \mathcal{G}$ generated by $e_1, ..., e_n \in E$, and for all $\varepsilon > 0$, there exists an automorphism $\phi : X \to X$ such that for each $1 \leq j \leq n$, $\|\phi \circ f_1(e_j) - f_2(e_j)\| < \varepsilon$.



In the Banach space setting, the embeddings are linear embeddings and the class of finitely generated spaces are finite dimensional spaces. In the lattice setting, the appropriate maps are isometric lattice embeddings, and one can either choose finite dimensional or finitely generated lattices.

The equimeasurability results described above can be used to show an approximate ultrahomogeneity of $L_p([0, 1])$ over its finite dimensional linear subspaces only so long as $p \notin 2\mathbb{N}$ (see [10]). Conversely, the cases where $p \in 2\mathbb{N}$ are not AUH over finite dimensional linear subspaces, with counterexamples showing linearly isometric spaces whose corresponding basis elements are not equimeasurability. Alternate methods using continuous Fraïssé Theory have since then been used to give alternate proofs of linear approximate ultrahomogeneity of L_p for $p \notin 2\mathbb{N}$ (see [5]) as well as lattice homogeneity of L_p for all $1 \leq p < \infty$ (see [2], [5]).

This paper is structured as follows: in section 2, we first establish basic notation and give a characterization of finite dimensional BL_pL_q lattices. This characterization is used in subsequent sections for establishing both equimeasurability and ultrahomogeneity results later on.

In section 3 we show that when $p \neq q$, $L_p(L_q) := L_p(L_q)$ is AUH over the larger class of finite dimensional (and finitely generated) BL_pL_q spaces. This is done by characterizing representations of BL_pL_q sublattices $L_p(L_q)$ in such a way that induces automorphisms over $L_p(L_q)$ making the homogeneity diagram commute. The results here play a role in subsequent sections as well.

In section 4, we prove that if in addition $p/q \notin \mathbb{N}$, $L_p(L_q)$ is also AUH over the class of its finitely generated sublattices. First, we determine the isometric structure of finite dimensional sublattices of $L_p(L_q)$ lattices by giving an alternate proof of [7, Proposition 3.2] showing that two sublattices E and F of $L_p(L_q)$, with the e_i's and f_i's each forming the basis of atoms, are lattice isometric iff $(N[e_1], ..., N[e_n])$ and $(N[f_1], ..., N[f_n])$ are equimeasurable. The equimeasurability result allows us to reduce a homogeneity diagram involving a finite dimensional sublattice of $L_p(L_q)$ to one with a finite dimensional BL_pL_q lattice, from which, in combination with the results in section 3, the main result follows.

Section 5 considers the case of $p/q \in \mathbb{N}$. Here, we provide a counterexample to equimeasurability in the case that $p/q \in \mathbb{N}$ and use this counterexample to show that in such cases, $L_p(L_q)$ is not AUH over the class of its finite dimensional lattices.

2. Preliminaries

We begin with some basic notation and definitions. Given a measurable set $A \subseteq \mathbb{R}^n$, we let $\mathbf{1}_A$ refer to the characteristic function over A. For a lattice X, let $B(X)$ be the unit ball, and $S(X)$ be the unit sphere.

For elements $e_1, ..., e_n$ in some lattice X, use bracket notation $< e_1, ..., e_n >_L$ to refer to the Banach lattice generated by the elements $e_1, ..., e_n$. In addition, we write $< e_1, ..., e_n >$ without the L subscript to denote that the generating elements e_i are also mutually disjoint positive elements in the unit sphere. Throughout, we will also use boldface notation to designate a finite sequence of elements: for instance, for $x_1, ..., x_n \in \mathbb{R}$ or $x_1, ..., x_n \in X$ for some lattice x, let $\mathbf{x} = (x_1, ..., x_n)$. Use the same notation to denote a sequence of functions over corresponding elements: for example, let $(f_1, ..., f_n) = \mathbf{f}$, or $(f_1(x_1), ...f_n(x_n)) = \mathbf{f}(\mathbf{x})$, or $(f(x_1), ..., f(x_n)) = f(\mathbf{x})$. Finally, for any element e or tuple $\mathbf{e}$ of elements in some lattice X, let $\boldsymbol{\beta}(e)$

and $\boldsymbol{\beta}(\mathbf{e})$ be the band generated by e and $\mathbf{e}$ in X, respectively.

Recall that Bochner integrable functions are the norm limits of simple functions $f : \Omega \to L_q(\Omega')$, with $f(\omega) = \sum_1^n r_i \mathbf{1}_{A_i}(\omega)\mathbf{1}_{B_i}$, where $\mathbf{1}_{A_i}$ and $\mathbf{1}_{B_i}$ are the characteristic functions for $A_i \in \Sigma$ and $B_i \in \Sigma'$, respectively. One can also consider $f \in L_p(\Omega; L_q(\Omega'))$ as a $\Sigma \otimes \Sigma'$-measurable function such that

$$\|f\| = \left(\int_\Omega \|f(\omega)\|_q^p \, d\omega \right)^{1/p} = \left(\int_\Omega \left(\int_{\Omega'} |f(\omega, \omega')|^q \, d\omega' \right)^{p/q} d\omega \right)^{1/p}$$

Unlike the more familiar L_p lattices, the class of abstract L_pL_q lattices is not itself axiomatizable; however, the slightly more general class BL_pL_q of bands in $L_p(L_q)$ lattices is axiomatizable. Additionally, if X is a separable BL_pL_q lattice, it is lattice isometric to a lattice of the form

$$\begin{aligned} &\left(\bigoplus_p L_p(\Omega_n; \ell_q^n) \right) \oplus_p L_p(\Omega_\infty; \ell_q) \\ \oplus_p &\left(\bigoplus_p L_p(\Omega_n'; L_q(0,1) \oplus_q \ell_q^n) \right) \\ \oplus_p \ & L_p(\Omega_\infty'; L_q(0,1) \oplus_q \ell_q). \end{aligned}$$

BL_pL_q lattices may also contain what are called *base disjoint* elements. x and y are base disjoint if $N[x] \perp N[y]$. Based on this, we call x a *base atom* if whenever $0 \le y, z \le x$ with y and z base disjoint, then either $N[y] = 0$ or $N[z] = 0$. Observe this implies that $N[x]$ is an atom in L_p. Alternatively, we call x a *fiber atom* if any disjoint $0 \le y, z \le x$ are also base disjoint. Finally, we say that X is *doubly atomless* if it contains neither base atoms nor fiber atoms.

Another representation of BL_pL_q involves its finite dimensional subspaces. We say that X is an $(\mathcal{L}_p\mathcal{L}_q)_\lambda$ lattice, with $\lambda \ge 1$ if for all disjoint $x_1, ..., x_n \in X$ and $\varepsilon > 0$, there is a finite dimensional F of X that is $(1+\varepsilon)$-isometric to a finite dimensional BL_pL_q space containing $x_1', ..., x_n'$ such that for each $1 \le i \le n$, $\|x_i - x_i'\| < \varepsilon$. Henson and Raynaud proved that in fact, any lattice X is a BL_pL_q space iff X is $(\mathcal{L}_p\mathcal{L}_q)_1$ (see [6]). This equivalence can be used to show the following:

Proposition 2.1. *(Henson, Raynaud) If X is a separable BL_pL_q lattice, then it is the inductive limit of finite dimensional BL_pL_q lattices.*

The latter statement is not explicitly in the statement of Lemma 3.5 in [6], but the proof showing that any BL_pL_q lattice is $(\mathcal{L}_p\mathcal{L}_q)_1$ was demonstrated by proving the statement itself.

Throughout this paper, we refer to this class of finite dimensional BL_pL_q lattices as $B\mathcal{K}_{p,q}$. Observe that if $E \in B\mathcal{K}_{p,q}$, then it is of the form $\oplus_p(\ell_q^{m_i})_1^N$ where for $1 \leq k \leq N$, the atoms $e(1,1), ..., e(k, m_k)$ generate $\ell_q^{m_k}$.

Proposition 2.2. *Let E be a $B\mathcal{K}_{p,q}$ sublattice of $L_p(L_q)$ with atoms $e(k,j)$ as described above. Then the following are true:*

(1) There exist disjoint measurable $A(k) \subseteq [0,1]$ such that for all i, $\text{supp}(e(k,j)) \subseteq A(k) \times [0,1]$,
(2) For all k and for all j, j', $N[e(k,j)] = N[e(k,j')]$.

Conversely, if E is a finite dimensional sublattice of $L_p(L_q)$ satisfying properties (1) and (2), then E is in $B\mathcal{K}_{p,q}$.

In order to prove this theorem, we first need the following lemma:

Lemma 2.3. *Let $0 < r < \infty$, with $r \neq 1$. suppose $x_1, ..., x_n \in L_r+$ are such that*

$$\|\sum_1^n x_k\|_r^r = \sum \|x_k\|_r^r$$

Then the x_i's are mutually disjoint.

Proof. If $r < 1$, then

$$\int x_i(t)^r + x_j(t)^r \; dt = \|x_i\|_r^r + \|x_j\|_r^r = \int (x_i(t) + x_j(t))^r \; dt \tag{1}$$

Now observe that for all t, $(x_i(t)+x_j(t))^r \leq x_i(t)^r + x_j(t)^r$, with equality iff either $x_i(t) = 0$ or $x_j(t) = 0$, so $(x_i + x_j)^r - x_i^r - x_j^r \in L_1+$. Combined with the above equality in line (1), since $\|(x_i + x_j)^r - x_i^r - x_j^r\|_1 = 0$, it follows that $x_i(t)^r + x_j(t)^r = (x_i(t) + x_j(t))^r$ a.e., so x_i must be disjoint from x_j when $i \neq j$.

If $r > 1$, proceed as in the proof for $r < 1$, but with the inequalities reversed, given that in this instance $x_i(t)^r + x_j(t)^r \leq (x_i(t) + x_j(t))^r$ for all t.

□

Remark 2.4. The above implies that a BL_pL_q lattice X is base atomless if it contains no bands lattice isometric to some L_p or L_q space. Indeed, if there were a base atom e, then any two $0 \leq x \perp y \leq e$ would have to have N-norms multiple to each other, so $< x, y >$ is lattice isometric to ℓ_q^2. Resultantly, the band generated by e is an L_q space. Similarly, if e is a fiber atom, then any $0 \leq x \perp y \leq e$ is also base disjoint, which implies that the band generated by e is an L_p space.

We now conclude with the proof of Proposition 2.2:

Proof of Proposition 2.2. Observe that for each appropriate pair (k, j),

$$\left(\int_0^1 N[e(k,j)]^p(s)\ ds\right)^{q/p} = \|N^q[e(k,j)]\|_{p/q} = 1$$

For notational ease, let $E(k,j) = N^q[e(k,j)]$. Pick $j_1, ..., j_n$ with each $j_k \leq m_k$. Then, by disjointness of the $e(k,j)$'s, for all $(a_k)_k \geq 0$ and all $x = \sum_k a_k e(k, j_k)$,

$$\begin{aligned}\|\sum a_k e(k,j_k)\|^q &= \left(\int_0^1 \left(\sum_k a_k^q E(k,j_k)(s)\right)^{p/q} ds\right)^{q/p} \\ &= \left\|\sum a_k^q E(k,j_k)\right\|_{p/q}.\end{aligned}$$

Now since the $e(k, j_k)$'s are isometric to ℓ_p,

$$\left\|\sum a_k^q E(k,j_k)\right\|_{p/q}^{p/q} = \sum_i a_k^p = \sum_k (a_k^q)^{p/q} = \sum_k \|a_k^q E(k,j_k)\|_{p/q}^{p/q}.$$

Since the $E(k, j_k)$'s are all positive and $p \neq q$, by Lemma 2.3, the $E(k, j_k)$'s are disjoint, that is, the $e(k, j_k)'s$ are base disjoint.

For $1 \leq k \leq N$, let $A(1), ..., A(n)$ be mutually disjoint measurable sets each supporting each $E(k,j)$ for $1 \leq j \leq n_k$. Then each $e(k,j)$ is supported by $A(k) \times [0,1]$. Now we prove (2). Fix k, Then using similar computations as above, and since the $e(k,j)$'s for fixed k generate $\ell_q^{m_k}$:

$$\|\sum_j a_j e(k,j)\|^q = \left\|\sum_j a_j^q E(k,j)\right\|_{p/q} = \sum_j a_j^q = \sum_j a_j^q \|E(k,j)\|_{p/q}$$

By Minkowski's inequality, as $p \neq q$, equality occurs only when $E(k,j)(s) = E(k,j')(s)$ a.e. for all $1 \leq j, j' \leq n_i$.

To show the converse, it is enough to give the computation:

$$\begin{aligned}\|\sum_{k,j} a(k,j)e(k,j)\| &= \left(\int_0^1 \left[\int \left(\sum_{k,j} a(k,j)e(k,j)(s,t)\right)^q dt\right]^{p/q} ds\right)^{1/p} \\ &= \left(\sum_k \int_0^1 \left[\sum_{j=1}^{n_i} |a(k,j)|^q E(k,j)(s)\right]^{p/q} ds\right)^{1/p} \\ &= \left(\sum_k \left[\sum_{j=1}^{n_k} |a(k,j)|^q\right]^{p/q} \int_0^1 E(k,1)^{p/q}(s)\ ds\right)^{1/p} \\ &= \left(\sum_k \left[\sum_{j=1}^{n_k} |a(k,j)|^q\right]^{p/q}\right)^{1/p}\end{aligned}$$

□

The following results will allow us to reduce homogeneity diagrams to those in which the atoms $e(k,j)$ of some $E \in B\mathcal{K}_{p,q}$ are mapped by both embeddings to characteristic functions of measurable $A(k,j) \subseteq [0,1]^2$. In fact, we can further simplify such diagrams to cases where E is generated by such $e(k,j)$'s which additionally are *base-simple*, i.e., $N[e(k,j)]$ is a simple function.

Proposition 2.5. *Let $1 \leq p \neq q < \infty$ and let $e \in S(L_p(L_q))_+$ be an element with full support over $[0,1]^2$. Then there exists a lattice automorphism ϕ from $L_p(L_q)$ to itself such that $\phi(\mathbf{1}) = e$. Furthermore, ϕ can be constructed to bijectively map both simple functions to simple functions and base-simple functions to base-simple functions.*

Proof. The proof is an expansion of the technique used in Lemma 3.3 from [5]. Given a function $g(y) \in L_{q_+}$, define $\tilde{g}(y)_q$ by $\tilde{g}(y)_q = \int_0^y g(t)^q \, dt$, and for notation, use $e_x(y) = e(x,y)$. Since e has full support, we may assume that for all $0 \leq x \leq 1$, $N[e](x) > 0$. From there, Define ϕ by

$$\phi(f)(x,y) = f\left(\widetilde{N[e]}(x)_p, \frac{\tilde{e}_x(y)_q}{N^q[e](x)}\right)e(x,y)$$

$e \geq 0$ and the rest of the function definition is a composition, so ϕ is a lattice homomorphism. To show it is also an isometry, simply compute the norm, using substitution in the appropriate places:

$$\begin{aligned}
\|\phi(f)\|^p &= \int_0^1 \left| \int_0^1 f\left(\widetilde{N[e]}(x)_p, \frac{\tilde{e}_x(y)_q}{N^q[e](x)}\right)^q e(x,y)^q \, dy \right|^{p/q} dx \\
&= \int_0^1 \left| \int_0^1 f(\widetilde{N[e]}(x)_p, y)^q \, dy \right|^{p/q} N^p[e](x) \, dx \\
&= \int_0^1 N[f](\widetilde{N[e]}(x)_p)^p N^p[e](x) \, dx \\
&= \int_0^1 N^p[f](x) \, dx = \|f\|^p.
\end{aligned}$$

To show surjectivity, let $B \subseteq [0,1]^2$ be a measurable set. Note that any $(x', y') \in [0,1]^2$ can be expressed as $(\widetilde{N[e]}(x)_p, \frac{\tilde{e}_x(y)_q}{N^q[e](x)})$ for some x, y, since $\widetilde{N[e]}(x)_p$ is an increasing continuous function from 0 to 1, while $\tilde{e}_x(y)_q$ is continuously increasing from 0 to $N^q[e](x)$. Thus there exists B' such that $\phi(\mathbf{1}_{B'}) = \mathbf{1}_B \cdot e$, implying that ϕ's image is dense in the band generated by $\boldsymbol{\beta}(e) = L_p(L_q)$ since e has full support. Therefore, ϕ is also surjective.

Finally, ϕ consists of function composition into f multiplied by e, so if e and f are simple, then it has a finite image, so if f is simple, then the product is also simple, ϕ maps simple functions to simple functions, Conversely, if $\phi(f)$ is simple, then $\phi(f)/e$ is also simple. Thus $f\left(\widetilde{N[e]}(x)_p, \frac{\tilde{e}_x(y)_q}{N[e](x)}\right)$ has a finite image. It follows that f itself has a finite image.

Using similar reasoning, if $N[e]$ is simple, then whenever $N[f]$ is simple, $N[\phi(f)]$ must also be simple, and likewise the converse is true, since by the computation above, $N[\phi(f)](x) = N[f](\widetilde{N[e]}(x)_p) \cdot N[e](x)$. □

3. Approximate Ultrahomogeneity of $L_p(L_q)$ over BL_pL_q spaces

In this section, we show that for any $1 \leq p \neq q < \infty$, $L_p(L_q)$ is AUH over $B\mathcal{K}_{p,q}$.

Let $\mathbf{f} := (f_1, ..., f_n)$ and $\mathbf{g} := (g_1, ..., g_n)$ be sequences of measurable functions and let λ be a measure in $\mathbb{R}$. Then we say that $\mathbf{f}$ and $\mathbf{g}$ are *equimeasurable* if for all λ-measurable $B \subseteq \mathbb{R}^n$,

$$\lambda(t : \mathbf{f}(t) \in B) = \lambda(t : \mathbf{g}(t) \in B)$$

We also say that functions $\mathbf{f}$ and $\mathbf{g}$ in $L_p(L_q)$ are *base-equimeasurable* if $N(\mathbf{f})$ and $N(\mathbf{g})$ are equimeasurable.

Lusky's main proof in [10] of linear approximate ultrahomogeneity in $L_p(0,1)$ for $p \neq 4, 6, 8, ...$ hinges on the equimeasurability of generating elements for two copies of some $E =< e_1, ..., e_n >$ in L_p containing $\mathbf{1}$. But when $p = 4, 6, 8, ...$, there exist finite dimensional E such that two linearly isometric copies of E in L_p do not have equimeasurable corresponding basis elements. However, if homogeneity properties are limited to E with mutually disjoint basis elements, then E is linearly isometric to ℓ_p^n, and for all $1 \leq p < \infty$, L_p is AUH over all ℓ_p^n spaces. Note that here, an equimeasurability principle (albeit a trivial one) also applies: Any two copies of $\ell_p^n =< e_1, ..., e_n >$ into $L_p(0,1)$ with $\sum_k e_k = n^{1/p} \cdot \mathbf{1}$ have (trivially) equimeasurable corresponding basis elements to each other as well.

In the $L_p(L_q)$ setting, similar results arise, except rather than comparing corresponding basis elements $f_i(e_1), ..., f_i(e_n)$ of isometric copies $f_i(E)$ of E, equimeasurability results hold in the L_q-norms $N[f_i(e_j)]$ under similar conditions, with finite dimensional BL_pL_q lattices taking on a role like ℓ_p^n does in L_p spaces.

The following shows that equimeasurability plays a strong role in the approximate ultrahomogeneity of $L_p(L_q)$ by showing that any automorphism fixing $\mathbf{1}$ preserves base-equimeasurability for characteristic functions:

Proposition 3.1. *Suppose $p \neq q$, and let $T : L_p(L_q)$ be a lattice automorphism with $T(\mathbf{1}) = \mathbf{1}$. Then there exists a function $\phi \in L_p(L_q)$ and a measure preserving transformation ψ over L_p such that for a.e. $x \in [0,1]$,*

$\phi(x,\cdot)$ is also a measure preserving transformation inducing an isometry over L_q, and for all f,

$$Tf(x,y) = f(\psi(x), \phi(x,y)).$$

Furthermore, for all measurable $B_1, ..., B_n \subseteq [0,1]^2$ with $\mathbf{1}_{B_i}$'s mutually disjoint, $(\mathbf{1}_{B_1}, ..., \mathbf{1}_{B_n})$ and $(T\mathbf{1}_{B_1}, ..., T\mathbf{1}_{B_n})$ are base-equimeasurable.

Proof. By the main result in [13], there exists a strongly measurable function $\Phi : [0,1] \to B(L_q)$, a set isomorphism Ψ over L_p (see [13] for a definition on set isomorphisms), and some $e(x) \in L_p$ related to the radon-Nikodym derivative of Ψ such that

$$Tf(x)(y) = \Phi(x)(e(x)\Psi f(x))(y),$$

and for a.e. x, $\Phi(x)$ is a linear isometry over L_q. Observe first that T sends any characteristic function $1_{A\times[0,1]} \in L_p(L_q)$ constant over y to characteristic function $\mathbf{1}_{\psi(A)\times[0,1]}$ for some $\psi(A) \subseteq [0,1]$, so since $1_{A\times[0,1]} \in L_p(L_q)$ is constant over y, we can just refer to it as $\mathbf{1}_A$. Also, since T is a lattice isometry, $\mu(A) = \mu(\psi(A))$, so ψ is measure preserving. Finally, observe that $N[\mathbf{1}_A] = \mathbf{1}_A$. Thus, for any simple function $g := \sum c_i \mathbf{1}_{A_i} \in L_p(L_q)_+$ constant over y with the A_i's mutually disjoint, we have $N[g] = g$, and $Tg = g'$. Then for all x,

$$N[g'](x) = N[Tg](x) = N[\Phi(x)(eg')](x) = e(x)N[\Phi(x)(g')][x] = |e(x)|N[g'](x)$$

It follows that $|e(x)| = 1$. We can thus adjust Φ by multiplying by -1 where $e(x) = -1$. Note also that Φ acts as a lattice isometry over L_p when restricted to elements constant over y, so by Banach's theorem in [1], the map $\Phi f(x)$ can be interpreted as $\Phi(x)(\ f(\psi(x))\ )$, where ψ is a measure preserving transformation over $[0,1]$ inducing Ψ. By Banach's theorem again for $\Phi(x)$, this Φ can be interpreted by $\Phi f(x,y) = e'(x,y)f(\psi(x), \phi(x,y))$, with $\phi(x,\cdot)$ a measure preserving transformation for a.e. x. But since $T\mathbf{1} = \mathbf{1}$, this $e'(x,y) = 1$ as well.

It remains to prove equimeasurability. Let $\mathbf{1_B} = (\mathbf{1}_{B_1}, ..., \mathbf{1}_{B_n})$, and observe that since for a.e. x, $\phi(x,\cdot)$ is a measure preserving transformation inducing a lattice isometry over L_q, it follows that

$$N^q[\mathbf{1}_{B_i}](x) = \mu(y : (x,y) \in B_i) = \mu(y : (x, \phi(x,y)) \in B_i),$$

While

$$\begin{aligned} N^q[T\mathbf{1}_{B_i}](x) &= \mu(y : (\psi(x), \phi(x,y)) \in B_i) \\ &= \mu(y : (\psi(x), y) \in B_i) = N^q[\mathbf{1}_{B_i}](\psi(x)). \end{aligned}$$

Thus for each $A = \prod_i A_i$ with $A_i \subseteq [0,1]$ measurable, since ψ is also a measure preserving transformation,

$$\mu(x : (N^q[\mathbf{1_B}](x) \in A) = \mu(x : (N^q[\mathbf{1_B}](\psi(x)) \in A) = \mu(x : (N^q[T\mathbf{1_B}](x) \in A),$$

and we are done.

□

The following theorem describes a comparable equimeasurability property of certain copies of L_pL_q in $L_p(L_q)$ for any $1 \leq p \neq q < \infty$:

Theorem 3.2. *Let $1 \leq p \neq q < \infty$, and suppose that $f_i : E \to L_p(L_q)$ are lattice embeddings with $E \in B\mathcal{K}_{p,q}$ generated by a (k,j)-indexed collection of atoms $\mathbf{e} := (e(k,j))_{k,j}$ with $1 \leq k \leq n$ and $1 \leq j \leq m_k$ as described in Proposition 2.2. Suppose also that $f(\sum_{k,j} e(k,j)) = \mathbf{1} \cdot \| \sum e(k,j)\|$. Then $(f_1(\mathbf{e}))$ and $(f_2(\mathbf{e}))$ are base-equimeasurable.*

Proof. Let $\eta = \| \sum_{k,j} e(k,j)\|$, and note first that each $\frac{1}{\eta} f_i(e(k,j))$ is of the form $\mathbf{1}_{A_i(k,j)}$ for some measurable $A_i(k,j) \subseteq [0,1]^2$. Second, $N^q[\mathbf{1}_{A_i(k,j)}](s) = \mu(A_i(k,j)(s))$ with $A_i(k,j)(s) \subseteq [0,1]$ measurable for a.e. s, so by Proposition 2.2, for each fixed k and each j, j', $\mu(A_i(k,j)(s)) = \mu(A_i(k,j')(s)) = \frac{1}{m_k}\mathbf{1}_{A_i(k)}(s)$ with $A_i(1), ..., A_i(n) \subseteq [0,1]$ almost disjoint. It follows that for each appropriate k, j, $\frac{1}{\eta} = \frac{1}{m_k^{1/q}}\mu(A_i(k))^{1/p}$, so $\mu(A_i(k)) = \left(\frac{m_k^{1/q}}{\eta}\right)^p$.

To show equimeasurability, observe that for a.e. t, we have $N^q[\mathbf{1}_{A_i(k,j)}](s) = \frac{1}{m_k}$ iff $s \in A_i(k)$, and 0 otherwise. Let $\mathbf{B} \subseteq \prod_k \mathbb{R}^{m_k}$ be a measurable set. Note then that any (k,j)-indexed sequence $(N[f_i(\mathbf{e})](s))$ is of the form $\mathbf{c^i_s} \in \prod_k \mathbb{R}^{m_k}$ with $c^i_s(k,j) = \left(\frac{1}{m_k}\right)^{1/q}$ for some unique k, and $c^i_s(k,j) = 0$ otherwise. It follows then that for some $I \subseteq 1, ..., n$,

$$\mu(s : \mathbf{c^i_s} \in \mathbf{B}) = \sum_{k\in I} \mu(A_i(k)) = \sum_{k \in I} \left(\frac{m_k^{1/q}}{\eta}\right).$$

Since the above holds independent of our choice of i, we are done.

□

Remark 3.3. The above proof shows much more than base-equimeasurability for copies of $B\mathcal{K}_{p,q}$ lattices in $L_p(L_q)$. Indeed, if $\mathbf{1} \in E =< (e(k,j))_{k,j} >$ with $E \in B\mathcal{K}_{p,q}$, then each atom is in fact base-simple, and $\sum e(k,j) = \eta \cdot \mathbf{1}$ where $\eta = (\sum_k m_k^{p/q})^{1/p}$. Furthermore, there exist measurable sets $A(1), ..., A(n)$ partitioning $[0,1]$ with $\mu(A(k)) = \frac{m_k^{p/q}}{\eta^p}$ such that $N[e(k,j)] = \frac{\eta}{m_k^{1/q}}\mathbf{1}_{A(k)}$. Based on this, we can come up with a "canonical" representation of E, with $e(k,j) \mapsto \eta \cdot \mathbf{1}_{W_k \times V_{k,j}}$, where

$$W_k = \Big[\sum_{l=1}^{k-1} \mu(A(l)), \sum_{l=1}^{k} \mu(A(l))\Big], \text{ and } V_{k,j} = \left[\frac{j-1}{m_k}, \frac{j}{m_k}\right].$$

This canonical representation will become relevant in later results.

Having characterized representations of lattice in $B\mathcal{K}_{p,q}$, we now move towards proving the AUH result. Before the final proof, we use the following perturbation lemma.

Lemma 3.4. *Let $f : E \to L_p(L_q)$ be a lattice embedding of a lattice $E =< e_1, ..., e_n >$. Then for all $\varepsilon > 0$, there exists an embedding $g : E \to L_p(L_q)$ such that $g(E)$ fully supports $L_p(L_q)$ and $\|f - g\| < \varepsilon$.*

Proof. Let $M_k = supp\big(N[f(e_k)]\big) \backslash supp\big(N[f(\sum_1^{n-1} e_k)]\big)$. For each e_k, we will construct e_k' disjoint from $f(E)$ with support in $M_k \times [0,1]$. Let M' be the elements in $[0,1]^2$ disjoint from $f(E)$. Starting with $n = 1$, Observe that M' can be partitioned by $M' \cap M_k \times [0,1] := M_k'$. Let

$$\eta_k(x,y) = \varepsilon^{1/q} \frac{N[f(e_k)](x)}{\mu(M_k'(x))^{1/q}} \mathbf{1}_{M_k'}(x,y).$$

When $\mu(M_k'(x)) = 0$, let $\eta_k(x,y) = 0$ as well. Now, let $g' : E \to L_p(L_q)$ be the lattice homomorphism induced by

$$g'(e_k) = (1-\varepsilon)^{1/q} f(e_k) \cdot \mathbf{1}_{M_k} + \eta_n + f(e_k) \cdot \mathbf{1}_{M_k^c}.$$

First, we show that g' is an embedding. Observe that for each k,

$$\begin{aligned} N^q[g'(e_k)](x) =& \int \eta_k^q(x,y) + (1-\varepsilon) f(e_k)^q(x,y)\, dy \\ =& \int \varepsilon \frac{N^q[f(e_k)](x)}{\mu(M_k'(x))} \cdot \mathbf{1}_{M_k'}(x,y) + (1-\varepsilon) f(e_k)^q(x,y)\, dy \\ =& \varepsilon N^q[f(e_k)](x) + (1-\varepsilon) \int f(e_k)^q(x,y)\, dy \\ =& \varepsilon N^q[f(e_k)](x) + (1-\varepsilon) N^q[f(e_k)](x) = N^q[f(e_k)](x). \end{aligned}$$

It easily follows that $g'(E)$ is in fact isometric to $f(E)$, and thus to E. Furthermore, for every k,

$$\begin{aligned} \|f(e_k) - g'(e_k)\| =& \|\mathbf{1}_{M_k}[(1-(1-\varepsilon)^{1/q}) f(e_k) + \eta_k]\| \\ \leq& (1-(1-\varepsilon)^{1/q}) + \varepsilon. \end{aligned}$$

The above can get arbitrarily small.

Now, if $supp(N(\sum e_k)) = [0,1]$, let $g = g'$, and we are done. Otherwise, let $\tilde{M} = \cup_k M_k$, and observe that $\sum g'(e_k)$ fully supports $L_p(\tilde{M}; L_q)$. Observe also that $L_p(L_q) = L_p(\tilde{M}; L_q) \oplus_p L_p(\tilde{M}^c; L_q)$. However, both $L_p(\tilde{M}; L_q)$ and $L_p(\tilde{M}^c; L_q)$ are lattice isometric to $L_p(L_q)$ itself. So there exists an isometric copy of E fully supporting $L_p(\tilde{M}^c; L_q)$. Let $e_1', ..., e_n' \in L_p(\tilde{M}^c; L_q)$ be the corresponding basic atoms of this copy, and let $g(e_i) = (1-\varepsilon^p)^{1/p} g'(e_i) + \varepsilon \cdot e_n'$. Then for $x \in E$,

$$\|g(x)\|^p = (1-\varepsilon)\|g'(x)\|^p + \varepsilon \|x\|^p = \|x\|^p.$$

Using similar reasoning as in the definition of g', one also gets $\|g - g'\| < (1-(1-\varepsilon)^{1/p}) + \varepsilon$, so g can also arbitrarily approximate f.

□

Observe that the lemma above allows us to reduce the approximate homogeneity question down to cases where the copies of a $B\mathcal{K}_{p,q}$ lattice fully support $L_p(L_q)$. Combined with Proposition 2.5, we can further reduce the possible scenarios to cases where for each i, $f_i(x) = \mathbf{1}$ for some $x \in E$. It turns out these reductions are sufficient for constructing a lattice automorphism that makes the homogeneity diagram commute as desired:

Theorem 3.5. *Suppose $1 \leq p \neq q < \infty$, and for $i = 1, 2$, let $f_i : E \to L_p(L_q)$ be a lattice embedding with $E :=< (e(k,j))_{k,j} >\in B\mathcal{K}_{p,q}$ and $1 \leq k \leq n$ and $1 \leq j \leq m_k$. Suppose also that each $f_i(E)$ fully supports $L_p(L_q)$. Then there exists a lattice automorphism ϕ over $L_p(L_q)$ such that $\phi \circ f_1 = f_2$.*

Proof. Let $\eta = \|\sum_{k,j} e(k,j)\|$; by Proposition 2.5, we can assume that for both i's, we have $f_i(\sum_{k,j} e(k,j)) = \eta \cdot \mathbf{1}$. For notation's sake, let $e_i(k,j) := f_i(e(k,j))$. By Proposition 2.2, for each i there exist mutually disjoint sets $A_i(1), ..., A_i(n)$ partitioning $[0,1]$ such that for each $1 \leq j \leq m_k$, $supp(N[e_i(k,j)]) = A_i(k)$. In addition, for the sets $A_i(k,1), ..., A_i(k,m_k)$, where $A_i(k,j) := supp(e_i(k,j))$, partition $A_i(k) \times [0,1]$. It follows also from the statements in Remark 3.3 that $\mu(A_1(k)) = \mu(A_2(k))$ for each k and $N^q[e_i(k,j)](x) = \frac{\eta^q}{m_k}\mathbf{1}_{A_i(k)}(x)$.

To prove the theorem, it is enough to generate lattice automorphisms ϕ^i mapping each band $\boldsymbol{\beta}(e_i(k,j))$ to a corresponding band $\boldsymbol{\beta}(\mathbf{1}_{W_k \times V_{k,j}})$ where W_k and $V_{k,j}$ are defined as in Remark 3.3, with $\mathbf{1}_{A_i(k,j)} \mapsto \mathbf{1}_{W_k \times V_{k,j}}$.

To this end, we make a modified version of the argument in [7, Proposition 2.6] and adopt the notation in Proposition 2.5: construct lattice isometries $\psi^i_{k,j}$ from $L_p(A_i(k)); L_q(V_{k,j}))$ to $\boldsymbol{\beta}(e^i_{k,j})$ with

$$\psi^i_{k,j}(f)(x,y) = f\left(x, \left(\widetilde{\mathbf{1}}_{A_i(k,j)}\right)_x(y)_q + \frac{j-1}{m_k}\right)\mathbf{1}_{A_i(k,j)}(x,y)$$

By similar reasoning as in the proof of Proposition 2.5, $\psi^i_{k,j}$ is a lattice embedding. Surjectivity follows as well. Indeed, since $N^q[\mathbf{1}_{A_i(k,j)}](x) = \frac{1}{m_k}$, for a.e. $x \in A_i(k)$ the function $\left(\widetilde{\mathbf{1}}_{A_i(k,j)}\right)_x(y)_q + \frac{j-1}{m_k}$ matches $[0,1]$ continuously to $V_{k,j}$ with $supp(e_i(k,j)(x,\cdot))$ mapped a.e. surjectively to $V_{k,j}$. So $\psi^i_{k,j}$'s image is dense in $\boldsymbol{\beta}(e_i(k,j))$.

Observe that $\psi^i_{k,j}$ also preserves the random norm N along the base (that is: $N[f] = N[\psi^i_{k,j}(f)]$. Resultantly, the function $\psi^i_k := \oplus_j \psi^i_{j,k}$ mapping $L_p(A_i(k), L_q(0,1))$ to $\oplus_j \boldsymbol{\beta}(e_i(k,j))$ is also a lattice automorphism. Indeed, for $f = \sum_1^{m_k} f_j$ with $f_j \in \boldsymbol{\beta}(e_i(k,j))$, one gets

$$\|\psi_k^i(f)\| = \Big\|N[\sum_j \psi_{k,j}^i(f_j)]\Big\|_p = \Big\|\big(\sum_j N^q[\psi_{k,j}^i(f_j)]\big)^{1/q}\Big\|_p$$
$$= \Big\|\big(\sum_j N^q[f_j]\big)^{1/q}\Big\|_p = \Big\|N[\sum_j f_j]\Big\|_p = \|f\|$$

Now let $\psi^i = \oplus_k \psi_k^i$, and observe that given $f = \sum_1^n f_k$ with $f_k \in L_p(A_i(k), L_q(0,1))$, since the f_k's are base disjoint, we have

$$\|\psi^i f\|^p = \sum_1^n \|\psi_k^i f_k\|^p = \sum_1^n \|f_k\|^p = \|f\|^p.$$

Thus ψ^i is a lattice automorphism over $L_p(L_q)$ mapping each $1_{A_i(k)\times V_{k,j}}$ to $\mathbf{1}_{A_i(k,j)}$.

Use [5, Lemma 3.3] to construct a lattice isometry $\rho_i : L_p \to L_p$ such that for each k, $\rho_i(\mathbf{1}_{W_k}) = \mathbf{1}_{A_i(k)}$. By [1, Ch. 11 Theorem 5.1] this isometry is induced by a measure preserving transformation $\bar{\rho}_i$ from [0,1] to itself such that $\rho^i(f)(x) = f(\bar{\rho}_i(x))$. It is easy to show that ρ_i induces a lattice isometry with $f(x,y) \mapsto f(\bar{\rho}_i(x), y)$. In particular, we have $N[\rho_i f](x) = N[f](\bar{\rho}_i(x))$, and $\rho_i(\mathbf{1}_{W_k\times V_{k,j}}) = \mathbf{1}_{A_i(k)\times V_{k,j}}$, now let $\phi^i(f) = (\psi^i \circ \rho^i)(f)$, and we are done.

□

Using the above, we can now show:

Theorem 3.6. *For $1 \leq p \neq q < \infty$, the lattice $L_p(L_q)$ is AUH for the class $B\mathcal{K}_{p,q}$.*

Proof. Let $f_i : E \to L_p(L_q)$ as required, and suppose $\varepsilon > 0$. use Lemma 3.4 to get copies E_i' of $f_i(E)$ fully supporting $L_p(L_q)$ such that for each atom $e_k \in E$ and corresponding atoms $e_k^i \in E_i'$, we have $\|f_i(e_k) - e_k^i\| < \varepsilon/2$. now use Theorem 3.5 to generate a lattice automorphism ϕ from $L_p(L_q)$ to itself such that $\phi(e_k^1) = e_k^2$. Then

$$\|\phi(f_1(e_k)) - f_2(e_k))\| \leq \|\phi(f_1(e_k) - e_k^1)\| + \|e_k^2 - f_2(e_k)\| < \varepsilon$$

□

Remark 3.7. Observe that the doubly atomless $L_p(L_q)$ space is unique among separable BL_pL_q spaces that are AUH over $B\mathcal{K}_{p,q}$. Indeed, this follows from the fact that such a space must be doubly atomless to begin with: let E be a one dimensional space generated by atom e and suppose X is not doubly atomless. Suppose also that E is embedded by some f_1 into a part of X supported by some L_p or L_q band, and on the other hand is embedded by some f_2 into $F := \ell_p^2(\ell_q^2)$ with $f_2(e)$ a unit in F. Then one cannot almost extend f_1 to some lattice embedding $g : F \to X$ with almost commutativity.

One can also expand this approximate ultrahomogeneity to separable sublattices with a weaker condition of almost commutativity in the diagram for generating elements: for any BL_pL_q sublattice E generated by elements $< e_1, ..., e_n >_L$, for any $\varepsilon > 0$, and for all lattice embedding pairs $f_i : E \to L_p(L_q)$, there exists a lattice automorphism $g : L_p(L_q) \to L_p(L_q)$ such that for all $j = 1, ..., n$, $\|g(f_2(e_j)) - f_1(e_j)\| < \varepsilon$.

Theorem 3.8. *For all $1 \leq p \neq q < \infty$, The lattice $L_p(L_q)$ is AUH for the class of finitely generated BL_pL_q lattices.*

Proof. Let $E =< e_1, ...e_n >_L$, and let $f_i : E \to L_p(L_q)$ be lattice embeddings. We can assume that $\|e_k\| \leq 1$ for each $1 \leq i \leq n$. By Proposition 2.1, E is the inductive limit of lattices in $B\mathcal{K}_{p,q}$. Given $\varepsilon > 0$, pick a $B\mathcal{K}_{p,q}$ lattice $E' =< e'_1, ..., e'_m >\subseteq E$ such that for each e_k, there is some $x_k \in B(E')$ such that $\|x_k - e_k\| < \frac{\varepsilon}{3}$. Each $f_i|_{E'}$ is an embedding into $L_p(L_q)$, so pick an automorphism ϕ over $L_p(L_q)$ such that $\|\phi \circ f_1|_{E'} - f_2|_{E'}\| < \frac{\varepsilon}{3}$. Then

$$\|\phi f_1(e_k) - f_2(e_k)\| \leq \|\phi f_1(e_k - x_k)\| + \|\phi f_1(x_k) - f_2(x_k)\| + \|f_2(x_k - e_k)\| < \varepsilon.$$

□

We can also expand homogeneity to include not just lattice embeddings but also disjointness preserving linear isometries, that is, if embeddings $f_i : E \to L_p(L_q)$ are not necessarily lattice homomorphisms but preserve disjointness, then there exists a disjointness preserving linear automorphism ϕ over $L_p(L_q)$ satisfying almost commutativity:

Corollary 3.9. *$L_p(L_q)$ is AUH over finitely generated sublattices in $BL_p(L_q)$ with disjointness preserving embeddings.*

Proof. Use the argument in [5, Proposition 3.2] to show that $L_p(L_q)$ is disjointness preserving AUH over $B\mathcal{K}_{p,q}$. From there, proceed as in the argument in Theorem 3.8 to extend homogeneity over $B\mathcal{K}_{p,q}$ to that over BL_pL_q. □

4. Approximate Ultrahomogeneity of $L_p(L_q)$ when $p/q \notin \mathbb{N}$

The above results largely focused approximate ultrahomogeneity over BL_pL_q lattices. What can be said, however, of *sublattices* of L_pL_q spaces? The answer to this question is split into two cases: first, the cases where $p/q \notin \mathbb{N}$, and the second is when $p/q \in \mathbb{N}$. We address the first case in this section. It turns out that if $p/q \notin \mathbb{N}$, then $L_p(L_q)$ is AUH for the class of its finitely generated sublattices. The argument involves certain equimeasurability properties of copies of fixed finite dimensional lattices in $L_p(L_q)$. Throughout, we will refer to the class of sublattices of spaces in $B\mathcal{K}_{p,q}$ as simply $\mathcal{K}_{p,q}$, and let $\overline{\mathcal{K}_{p,q}}$ be the class of finitely generated sublattices of $L_p(L_q)$.

The following result appeared as [7, Proposition 3.2], which is a multi-dimensional version based on Raynaud's proof for the case of $n = 1$ (see [11, lemma 18]). The approach taken here is a multi-dimensional version of the proof of Lemma 2 in [8].

Theorem 4.1. *Let $r = p/q \notin \mathbb{N}$, and suppose $f_i : E \to L_p(L_q)$ are lattice isometric embeddings with $E =< e_1, ..., e_n >$. Suppose also that $f_1(x) = f_2(x) = \mathbf{1}$ for some $x \in E_+$. Then $f_1(\mathbf{e})$ and $f_2(\mathbf{e})$ are base-equimeasurable.*

Throughout the proof, let μ be a measure in some interval $I^n \subseteq C := \mathbb{R}^n_+$. To this end, we first show the following:

Lemma 4.2. *Suppose $0 < r \notin \mathbb{N}$, and α, β are positive finite Borel measures on C such that for all $\mathbf{v} \in C$ with $v_0 > 0$,*

$$\int_C |v_0 + \mathbf{v} \cdot \mathbf{z}|^r \ d\alpha(\mathbf{z}) = \int_C |v_0 + \mathbf{v} \cdot \mathbf{z}|^r \ d\beta(\mathbf{z}) < \infty.$$

Then $\alpha = \beta$.

Proof. It is equivalent to prove that the signed measure $\nu := \alpha - \beta = 0$. First, observe that since $|\nu| \leq \alpha + \beta$, and for any $\mathbf{v} \geq 0$, $\int |v_0 + \mathbf{v} \cdot \mathbf{z}|^r \ d|\nu|(\mathbf{z}) < \infty$.

Now, we show by induction on polynomial degree that for all $k \in \mathbb{N}$, $\mathbf{v} \geq 0$, and for all multivariate polynomials $P(\mathbf{z})$ of degree $k' \leq k$,

$$* \int_C |v_0 + \mathbf{v} \cdot \mathbf{z}|^{r-k} P(\mathbf{z}) \ d\nu(\mathbf{z}) = 0.$$

This is true for the base case $k = 0$ by assumption. Now assume it is true for $k \in \mathbb{N}$ and let $k' := \sum l_i \leq k$ with $\mathbf{l} \in \mathbb{N}^n$. For notational ease, let $\mathbf{z}^\mathbf{l} = z_1^{l_1} ... z_n^{l_n}$. Then for each v_i and $0 < t < 1$,

$$\int_{R^n_+} \mathbf{z}^\mathbf{l} \frac{(v_0 + \mathbf{v} \cdot \mathbf{z} + z_i t)^{r-k} - (v_0 + \mathbf{v} \cdot \mathbf{z})^{r-k}}{t} \ d\nu(\mathbf{z}) = 0.$$

Now, if $k + 1 < r$ and $t \in (0, 1)$, then

$$\left| \mathbf{z}^\mathbf{l} \frac{(v_0 + \mathbf{v} \cdot \mathbf{z} + z_i t)^{r-k} - (v_0 + \mathbf{v} \cdot \mathbf{z})^{r-k}}{t} \right| \leq \mathbf{z}^\mathbf{l} z_i (r-k)(v_0 + \mathbf{v} \cdot \mathbf{z} + v_i)^{r-k-1}$$
$$\leq \frac{r-k}{\mathbf{v}^\mathbf{l} v_i} (v_0 + \mathbf{v} \cdot \mathbf{z} + v_i)^r$$

Since in this case, $0 < r - k - 1 < r$ and $|\nu| < \infty$, the right hand side must also be $|\nu|$-integrable. On the other hand, If $k + 1 > r$, then we have

$$\left| \mathbf{z}^\mathbf{l} \frac{(v_0 + \mathbf{v} \cdot \mathbf{z} + v_i t)^{r-k} - (v_0 + \mathbf{v} \cdot \mathbf{z})^{r-k}}{t} \right| < |r - k| \frac{v_0^r}{\mathbf{v}^\mathbf{l} v_i}$$

which is also $|\nu|$-integrable. So now we apply Lebesgue's differentiation theorem over v_i to get, for any $k \in \mathbb{N}$ and for each $1 \leq i \leq n$:

$$\int_C \mathbf{z}^{\mathbf{l}} z_i |v_0 + \mathbf{v}\cdot\mathbf{z}|^{r-k-1}\ d\nu(\mathbf{z}) = 0,$$

since $r \notin \mathbb{N}$. A similar argument, deriving over v_0, can be made to show that

$$\int_C |v_0 + \mathbf{v}\cdot\mathbf{z}|^{r-k-1}\ d\nu(\mathbf{z}) = 0$$

One can make linear combinations of the above, which implies line $*$.

Now for fixed $\mathbf{v} > 0$, $v_0 > 0$ we define a measure Λ on C, where for measurable $B \subseteq \mathbb{R}^n_+$,

$$\Lambda(B) = \int_{\phi^{-1}(B)} |v_0 + \mathbf{v}\cdot\mathbf{z}|^r\ d\nu(\mathbf{z}).$$

where $\phi(\mathbf{z}) = \frac{1}{v_0+\mathbf{v}\cdot\mathbf{z}}\mathbf{z}$. It is sufficient to show that $\Lambda = 0$. Observe first that ϕ is continuous and injective; indeed, if $\phi(\mathbf{z}) = \phi(\mathbf{w})$, then it can be shown that $\mathbf{v}\cdot\mathbf{w} = \mathbf{v}\cdot\mathbf{z}$. Thus $\frac{\mathbf{w}}{v_0+\mathbf{v}\cdot\mathbf{z}} = \frac{\mathbf{z}}{v_0+\mathbf{v}\cdot\mathbf{z}}$, implying that $\mathbf{w} = \mathbf{z}$. Resultantly, $\phi(B)$ for any Borel B is also Borel, hence we will have shown that for any such B, $\nu(B) = 0$ as well, so $\nu = 0$.

Observe that by choice of $\mathbf{v} > 0$ and and since $(v_0 + \mathbf{v}\cdot\mathbf{z}) > 0$ for all $\mathbf{z} \in \mathbb{R}^n_+$, have

$$|\Lambda|(B) = \int_{\phi^{-1}(B)} |v_0 + \mathbf{v}\cdot\mathbf{z}|^r\ d|\nu|(\mathbf{z}).$$

Using simple functions and the definition of Λ, one can show both that for each i, we have

$$** \quad m_i(k) := \int_C w_i^k\ d|\Lambda|(\mathbf{w}) = \int_C (v_0 + \mathbf{v}\cdot\mathbf{z})^{r-k} z_i^k\ d\|\nu|(\mathbf{z}) < \infty$$

and also that

$$\int_C w_i^k\ d\Lambda(\mathbf{w}) = \int_C |v_0 + \mathbf{v}\cdot\mathbf{z}|^{r-k} z_i^k\ d\nu(\mathbf{z}) = 0,$$

More generally, if $k = \sum_i l_i$, then

$$\int_C \mathbf{w}^{\mathbf{l}}\ d\Lambda(\mathbf{w}) = \int_C \mathbf{z}^{\mathbf{l}} |v_0 + \mathbf{v}\cdot\mathbf{z}|^{r-k}\ d\nu(\mathbf{z}) = 0,$$

So it follows that $\int_C P(\mathbf{w})\ d\Lambda(\mathbf{w}) = 0$ for all polynomials $P(\mathbf{w})$.

Now if $k > r$ and $\nu \neq 0$, since $v_i > 0$, we then we have

$$\begin{aligned} m_i(k) &= \int_C |v_0 + \mathbf{v}\cdot\mathbf{z}|^{r-k} z_i^k \ d\|\nu|(\mathbf{z}) \\ &\leq \int_C |v_0 + \mathbf{v}\cdot\mathbf{z}|^{r} v_i^{-k} \ d|\nu|(\mathbf{z}) \leq v_i^{-k}|\Lambda|(C) < \infty \end{aligned}$$

so

$$m_i(k)^{-1/2k} \geq v_i^{1/2}|\Lambda|(C)^{-1/2k}$$

Thus for each $1 \leq i \leq n$, $\sum_k m_i(k)^{-1/2k} = 0$. So by [4, Theorem 5.2], $|\Lambda|$ is the unique positive measure over C with moment values $m_i(k)$. Since $|\Lambda| + \Lambda$ yields the same values, and by **, $\int_C P(\mathbf{w})\ d(|\Lambda| + \Lambda)(\mathbf{w}) = \int_C P(\mathbf{w})\ d|\Lambda|(\mathbf{w})$, it follows that $\Lambda = 0$, so $\nu = 0$.

□

Now we are ready to prove Theorem 4.1.

Proof. For simplicity of notation, let $F_j^i = N^q[f_i(e_j)]$ and $I = [0,1]$. By definition of N, the support of F_j^i as well as of μ is the unit interval. Define positive measures α_j by

$$\alpha_i(B) = \mu(\{t \in I : \mathbf{F}^i(t) \in B\}) = \mu((\mathbf{F}^i)^{-1}(B)).$$

Now, for any measurable $B \subseteq C$, we have

$$\int_C \mathbf{1}_B(\mathbf{z})\ da_i(\mathbf{z}) = \alpha_i(B) = \mu((\mathbf{F}^i)^{-1}(B)) = \int_I (\mathbf{1}_B \circ \mathbf{F}^i)(t)\ dt$$

so for any simple function σ over C,

$$\int_C \sigma(\mathbf{z})\ d\alpha_i = \int_0^1 \sigma \circ \mathbf{F}^i(t)\ dt$$

Using simple functions to approximate $|v_0+\mathbf{v}\cdot\mathbf{z}|^r$, and given that $|v_0+\mathbf{v}\cdot\mathbf{z}|^r$ is in $L_1(C,\mu)$ and the support of μ is the unit interval, it follows that

$$\int_C |1 + \mathbf{v}\cdot\mathbf{z}|^r\ d\alpha_i(\mathbf{z}) = \int_0^1 |1 + \mathbf{v}\cdot\mathbf{F}^i(t)|^r\ dt.$$

It is sufficient now to show that for all $\mathbf{v} \in \mathbb{R}_+^n$,

$$\int_0^1 |1 + \mathbf{v}\cdot\mathbf{F}^1(t)|^r\ dt = \int_0^1 |1 + \mathbf{v}\cdot\mathbf{F}^2(t)|^r\ dt.$$

.

For i, j and $s \in [0,1]$, let $M_i^j = \{(s,t) : (s,t) \in \mathit{supp}(f_i(e_j))\}$, and let $M_i^j(s) = \{t : (s,t) \in M_i^j\}$. By assumption, $x = \sum_j x_j e_j$ with $x_j > 0$, so $\mathbf{1} = N^q[f_i(x)] = \sum_j x_j^q F_j^i$. Therefore, since each f_i is an embedding, for all $\mathbf{c} \in \mathbb{R}_+^n$,

$$\begin{aligned}\|\sum_j c_j e_j\|^p =& \Big\| \big(\sum_j c_j^q F_j^i(s)\big)^{1/q} \Big\|_p \\ =& \Big\| \big(\mathbf{1} + \sum_j (c_j^q - x_j^q) F_j^i(s)\big)^{1/q} \Big\|_p\end{aligned}$$

Let $v_j := c_j^q - x_j^q$: then in particular it follows that for all $\mathbf{v} \geq 0$, we have

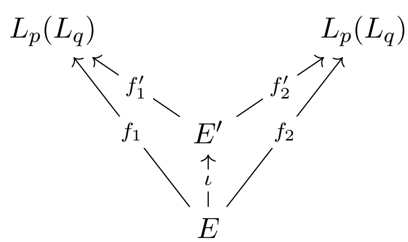
$$\int_0^1 \Big(1 + \mathbf{v} \cdot \mathbf{F}^1(s)\Big)^{p/q} ds = \int_0^1 \Big(1 + \mathbf{v} \cdot \mathbf{F}^2(s)\Big)^{p/q} ds.$$

By Lemma 4.2, we can conclude that $\alpha_1 = \alpha_2$, so $\mathbf{F}^1$ and $\mathbf{F}^2$ are equimeasurable. □

Using Theorem 4.1, we can uniquely characterize lattices in $\mathcal{K}_{p,q}$ in a way that parallels Proposition 2.2.

Theorem 4.3. *Suppose that $p/q \notin \mathbb{N}$, and let $E \subseteq L_p(L_q)$ with $E =< e_1, ..., e_m >$. Then the following hold:*

- *$E \in \mathcal{K}_{p,q}$ iff there exist mutually disjoint measurable functions $\phi(k,j) \in S(L_p(L_q))_+$, with $1 \leq j \leq n$ and $1 \leq k \leq L$ such that for each j, $e_j \in< (\phi(k,j))_k >= \ell_p^n$, and $< (\phi(k,j))_{k,j} >\in B\mathcal{K}_{p,q}$.*
- *Suppose $f_i : E \to L_p(L_q)$ is a lattice embedding with $i = 1, 2$ and $E \in \mathcal{K}_{p,q}$. Then there exist embeddings $f_i' : E' \to L_p(L_q)$ extending f_i such that $E' \in B\mathcal{K}_{p,q}$.*



Proof. For part 1, clearly the reverse direction is true. To prove the main direction, we can suppose that E fully supports $L_p(L_q)$. If not, recall that the band generated by E is itself doubly atomless, and hence is lattice isometric to $L_p(L_q)$ itself. Thus, if under these conditions, there is a BL_pL_q sublattices extending E as in the statement of the theorem, it will also be the case in general.

By Proposition 2.5, we can also suppose that $\sum_j e_j = \eta \cdot \mathbf{1}$. Now by assumption, since $E \in \mathcal{K}_{p,q}$, then there is an embedding $\psi : E \to \widetilde{E} \in B\mathcal{K}_{p,q}$ such that each $\psi(e_j) = \sum_k x(k,j)\tilde{e}(k,j)$, with $1 \leq k \leq m_k'$. Without loss of generality we may also drop any $\tilde{e}(k,j)$'s disjoint from $\psi(E)$ and assume that $\psi(E)$ fully supports $\widetilde{E}$. Now $\widetilde{E}$ is a $B\mathcal{K}_{p,q}$ lattice admitting a canonical representation in $L_p(L_q)$ as described in Theorem 3.2 and Remark 3.3.

So we can assume that ψ embeds E into $L_p(L_q)$ in such a way that $\psi(E)$ fully supports it and each $\psi(e_j)$ is both simple and base-simple. Now, use Proposition 2.5 to adjust ψ into an automorphism over $L_p(L_q)$ such that $\psi(\sum e_j) = \eta \cdot \mathbf{1}$ in a way that preserves both simplicity and base-simplicity. By Theorem 4.1, $\psi(\mathbf{e})$ and $\mathbf{e}$ are base-equimeasurable. Since the $\psi(k,j)'s$ are base-simple, there exist tuples $\mathbf{s^1}, ..., \mathbf{s^L} \in \mathbb{R}^m$ such that for a.e. $t \in [0,1]$, there is some $k \leq L$ such that $N[\mathbf{e}](t) = \mathbf{s^k}$. By equimeasurability, the same is true for $N[\psi(\mathbf{e})](t)$.

Let $\mathbf{S^k} = \{t : N[\mathbf{e}](t) = \mathbf{s^k}\}$, and let $S_j^k = \mathbf{S^k} \times [0,1] \cap supp(e_j)$. Let $\overline{\mathbf{S}}^{\mathbf{k}} = \{t : N[\psi(\mathbf{e})](t) = \mathbf{s^k}\}$ with $\overline{S}_j^k$ defined similarly. Note that each $\mathbf{1}_{S_j^k}$ is also base-characteristic, as $N[\mathbf{1}_{S_j^k}] = c_j^k \mathbf{1}_{\mathbf{S^k}}$ for some $c_j^k > 0$, so for fixed k and for any $j, j' \leq m_k$, we must have that $N[\mathbf{1}_{S_j^k}]$ and $N[\mathbf{1}_{S_{j'}^k}]$ are scalar multiples of each other. Thus for each appropriate pair (k,j) with $s_j^k > 0$, define $\phi(k,j)$ by $\frac{\mathbf{1}_{S_j^k}}{\|\mathbf{1}_{S_j^k}\|}$. By definition of $\mathbf{S^k}$, for any $k \neq k'$ and any appropriate j, j', $\phi(k,j)$ and $\phi(k',j')$ are fiber-disjoint, and $N[\phi(k,j)] = N[\phi(k,j')]$. Thus by Proposition 2.2, $< (\phi(k,j))_{k,j} > \in B\mathcal{K}_{p,q}$.

To prove part 2, Observe first that we have already essentially proven part 2 in the case that $f_1 = Id$ and $f_2 = \psi$. To show the general case, we first assume that for each i, $\sum f_i(e_j)$ maps to $\mathbf{1}$. Now, by Theorem 4.1, $f_1(\mathbf{e})$ and $f_2(\mathbf{e})$ are also base-equimeasurable, but by the procedure for part 1, we also know that each $f_i(e_j)$ is also base-simple. Define $\mathbf{s^1}, ..., \mathbf{s^L}$ as above, and Let $\mathbf{S^k}(i) = \{t : N[f_i(\mathbf{e})](t) = \mathbf{s^k}\}$. Define similarly $S_j^k(i)$ and the associated characteristic functions $\phi_i(k,j)$ for appropriate pairs k, j such that $1 \leq k \leq l$ and $s_j^k := \|\phi_i(k,j) \wedge f_i(e_j)\| > 0$. Note first that

$$f_i(e_j) = \sum_{k: s_k(j) > 0} s_j^k \phi_i(k,j).$$

Second, observe that by equimeasurability, the eligible pairs (k,j) are the same for $i = 1,2$. Let $E_i' =< (\phi_i(k,j))_{k,j} >$. Clearly $E_i' \in B\mathcal{K}_{p,q}$, and since the eligible pairs (k,j) are the same, E_1' and E_2' are isometric to each other. Let E' be one of the E_i''s and let $f_i' : E' \to L_p(L_q)$ be the expected embedding mapping E' to E_i', and we are done. $\square$

From here, we can now easily extend Theorem 3.5 to lattices in $\mathcal{K}_{p,q}$:

Corollary 4.4. *Suppose $p/q \notin \mathbb{N}$ and suppose $f_i : E \to L_p(L_q)$ are lattice embeddings from $E \in K_{p,q}$ with $f_i(E)$ fully supporting $L_p(L_q)$. Then there exists a lattice automorphism ϕ over $L_p(L_q)$ such that $f_2 = \phi \circ f_1$.*

Proof. Use Theorem 4.3 to generate a $B\mathcal{K}_{p,q}$ lattice E' containing E and lattice embeddings $f_i' : E' \to L_p(L_q)$ such that $f_i'|_E = f_i$. Clearly each $f_i'(E')$ fully supports $L_p(L_q)$. Now apply Theorem 3.5 to generate an automorphism ϕ over $L_p(L_q)$ with $\phi \circ f_1' = f_2'$. Clearly $\phi \circ f_1 = f_2$ as well.

□

When $p/q \notin \mathbb{N}$, using Theorem 4.3, we can show that the same holds with the more general class $\mathcal{K}_{p,q}$. However, we can make an even stronger claim by showing that homogeneity holds for any finite dimensional sublattice of $L_p(L_q)$. This is done using the following result, which gives a standard way of approximating finite dimensional sublattices of $L_p(L_q)$ with lattices in $\mathcal{K}_{p,q}$.

Lemma 4.5. *Suppose $p/q \notin \mathbb{N}$, and let $f_i : E \to L_p(L_q)$ be embeddings with $E =< e_1, ..., e_n >$. Then for all $\varepsilon > 0$, there exists a $\mathcal{K}_{p,q}$ lattice $E' =< e_1', ..., e_n' >$ and embeddings $g_i : E' \to L_p(L_q)$ such $g_i(E')$ fully supports $L_p(L_q)$ and for each n, $\|f_i(e_n) - g_i(e_n')\| < \varepsilon$.*

Proof. We can assume each $f_i(E)$ fully supports $L_p(L_q)$: given $\varepsilon > 0$, use Lemma 3.4 to get copies of E sufficiently close to each $f_i(E)$ with full support. We then also assume that $f_i(\sum_1^n e_k) = \mathbf{1}$ using Proposition 2.5.

By Theorem 4.1, $f_1(\mathbf{e})$ and $f_2(\mathbf{e})$ are base-equimeasurable. In particular, given any measurable $C \in \mathbb{R}^n$, one has $\mu(t : N[f_1(\mathbf{e})](t) \in C) = \mu(t : N[f_2(\mathbf{e})](t) \in C)$. Now pick an almost disjoint partition $C_1, ..., C_m$ of $S(\ell_1^n)$, where each C_i is closed, has relatively non-empty interior, and is of diameter less than $\frac{\varepsilon}{2n}$. Let $D_k^i = \{t : N[f_i(\mathbf{e})](t) \in C_i \setminus \cup_j^{i-1} C_j\}$. Then by equimeasurability, $\mu(D_k^1) = \mu(D_k^2)$. For each k, pick some $\mathbf{s^k} = (s_1^k, ..., s_n^k) \in C_k$, and for each $x \in D_k^i$, let

$$\overline{e}_j^i(x, y) = \frac{s_j^k}{N[f_i(e_j)](x)} f_i(e_j)(x, y).$$

Observe that $\| \sum_j \overline{e}_j^i - \sum_j f_i(e_j)\| < \varepsilon$, and $N[\overline{e}_j^i](x) = s_j^k$ for $x \in D_k^i$.

Consider now the lattice $E' =< \overline{e}_j^1, ..., \overline{e}_n^1 >$. Now, for any linear combination $\sum a_j \overline{e}_j^i$, we have, as in the argument in Proposition 2.5, that

$$\| \sum a_j \overline{e}_j^i \|^p = \sum_k^M (\sum_j (a_j s_j^k)^q)^{p/q}$$

implying that $\| \sum a_j \overline{e}_j^1 \| = \| \sum a_j \overline{e}_j^2 \|$. It follows both that E' embeds into $\ell_p^M(\ell_q^n)$, implying that it is a $\mathcal{K}_{p,q}$ lattice, and it is isometric to the lattice generated by the $\overline{e}_j^2$'s. Let $e_j' = \overline{e}_j^1$, and define $g_i : E' \to L_p(L_q)$ as the maps generated by $g_i(e_j') = \overline{e}_j^i$. Clearly these are lattice embeddings and $\|f_i(e_j) - g_i(e_j')\| < \varepsilon$.

□

Theorem 4.6. *For all $1 \leq p, q < \infty$ with $p/q \notin \mathbb{N}$, the lattice $L_p(L_q)$ is AUH for the class of finite dimensional sublattices of L_pL_q lattices.*

Proof. It is sufficient to show that the result is true over generation by basic atoms. Let $f_i : E \to L_p(L_q)$ be two embeddings with $E =< e_1, ..., e_n >$. Use Lemma 4.5 to find $g_i : E' \to L_p(L_q)$, with $E' :=< e'_1, ..., e'_n > \in \mathcal{K}_{p,q}$, $\|g_i(e'_k) - f_i(e_k)\| < \varepsilon/2$, and each $g_i(E')$ fully supporting $L_p(L_q)$. Then by Lemma 4.4, there exists an automorphism $\phi : L_p(L_q) \to L_p(L_q)$ such that $\phi \circ g_1 = g_2$. Note then that $\|\phi(f_1(e_k)) - f_2(e_k)\| \leq \|\phi(f_1(e_k) - g_1(e'_k))\| + \|f_2(e_k) - g_2(e'_k)\| < \varepsilon$. □

In a manner similar to that of Theorem 3.8, we can also extend the AUH property to finitely generated sublattices of $L_p(L_q)$ as well:

Theorem 4.7. *For all $1 \leq p, q < \infty$ with $p/q \notin \mathbb{N}$, The lattice $L_p(L_q)$ is AUH for the class $\overline{\mathcal{K}_{p,q}}$ of its finitely generated lattices.*

Proof. Suppose $E \subseteq L_p(L_q)$ is finitely generated. Then since E is order continuous and separable, it is the inductive limit of finite dimensional lattices as well, so pick a finite dimensional E' with elements sufficiently approximating the generating elements of E, and proceed with the same proof as in Theorem 3.8. □

The argument used in Corollary 3.9 can also be used to show:

Corollary 4.8. *For $p/q \notin \mathbb{N}$, $L_p(L_q)$ is disjointness preserving AUH over $\overline{\mathcal{K}_{p,q}}$.*

Remark 4.9. $L_p(L_q)$ for $p/q \notin \mathbb{N}$ is AUH over the entire class of its finitely generated sublattices, a property which is equivalent to such a class being a metric *Fraïssé class* with $L_p(L_q)$ as its *Fraïssé limit*. Recall that a class $\mathcal{K}$ of finitely generated lattices is *Fraïssé* if it satisfies the following properties:

(1) *Hereditary Property* (HP): $\mathcal{K}$ is closed under finitely generated sublattices.
(2) *Joint Embedding Property* (JEP): any two lattices in $\mathcal{K}$ lattice embed into a third in $\mathcal{K}$.
(3) *Continuity Property* (CP): any lattice operation symbol are continuous with respect to the Fraïssé pseudo-metric $d^{\mathcal{K}}$ in [2, Definition 2.11].
(4) *Near Amalgamation Property* (NAP): for any lattices $E =< e_1, ...e_n >_L$, F_1 and F_2 in $\mathcal{K}$ with lattice embeddings $f_i : E \to F_i$, and for all $\varepsilon > 0$, there exists a $G \in \mathcal{K}$ and embeddings $g_i : F_i \to G$ such that $\|g_1 \circ f_1(e_k) - g_2 \circ f_2(e_k)\| < \varepsilon$.
(5) *Polish Property* (PP): The Fraïssé pseudo-metric $d^{\mathcal{K}}$ is separable and complete in $\mathcal{K}_n$ (the $\mathcal{K}$-structures generated by n many elements).

Now clearly the finitely generated sublattices of $L_p(L_q)$ fulfill the first two properties, and the third follows from the lattice and linear operations having moduli of continuity independent of lattice geometry. In addition, if one can show that the class $\mathcal{K}$ has the NAP, has some separable X which is universal for $\mathcal{K}$, and its NAP amalgamate lattices can be chosen so that they are closed under inductive limits, then one can prove that $\mathcal{K}$ also has the Polish Property (a technique demonstrated in [14, Theorem 4.1] and more generally described in Section 2.5 of [9]). The main difficulty in proving that a class of lattices $\mathcal{K}$ is a Fraïssé class is in showing that it has the NAP. However, thanks to Theorem 4.7, we have

Corollary 4.10. *$\overline{\mathcal{K}_{p,q}}$ has the NAP.*

Theorem 4.7 implies an additional collection of AUH Banach lattices to the currently known AUH Banach lattices: namely L_p for $1 \leq p < \infty$,the Gurarij M-space $\mathcal{M}$ discovered in [5], and the Gurarij lattice discovered in [14].

However, if one considers classes of finite dimensional Banach spaces with Fraïssé limits using linear instead of lattice embeddings, the only known separable AUH Banach spaces are the Gurarij space and L_p for $p \neq 4, 6, 8, ...$, and it is currently unknown if there are other Banach spaces that are AUH over its finite dimensional subspaces with linear embeddings. Certain combinations of p and q are also ruled out for $L_p(L_q)$ as a potential AUH candidate as discussed in Problem 2.9 of [5]: in particular, when $1 \leq p, q < 2$, $L_p(L_q)$ cannot be linearly AUH.

5. Failure of homogeneity for $p/q \in \mathbb{N}$

Recall that when $E =< e_1, ..., e_n >\in B\mathcal{K}_{p,q}$ is embedded into $L_p(L_q)$ through f_1, f_2, then we can achieve almost commutativity for any $p \neq q$. However, the automorphism in Theorem 3.6 clearly preserves the equimeasurability of the generating basic atoms of $f_i(E)$ as it fixes $\mathbf{1}$.

In this section, we show that the results of Section 4 do not hold when $p/q \in \mathbb{N}$. The first results in this section show that when some $e \in L_p(L_q)_+$ is sufficiently close to $\mathbf{1}$, the automorphism originally used in the argument of Proposition 2.5 sending $\mathbf{1}$ to e also perturbs selected functions piecewise continuous on their support in a controlled way. Second, Theorem 4.1 does not hold, and thus we cannot infer equimeasurability for arbitrary finite dimensional sublattices of $L_p(L_q)$. Finally, we use these results to strengthen the homogeneity property for any $L_p(L_q)$ lattice assumed to be AUH, and then show that when $p/q \in \mathbb{N}$, $L_p(L_q)$ does not fulfill this stronger homogeneity property, and thus cannot be AUH.

Lemma 5.1. *Let $1 \leq p \neq q < \infty$, and let $< f_1, ..., f_n >\subseteq L_p(L_q)$ be such that $\sum f_i = \mathbf{1}$. Suppose also that for a.e. x, $f_k(x, \cdot) = \mathbf{1}_{[g_k(x), g_{k+1}(x)]}$ where each g_k has finitely many discontinuities. Let $\varepsilon > 0$, and let $e \in S(L_p(L_q))_+$*

fully support $L_p(L_q)$. Consider

$$\phi(f)(x,y) = f\left(\widetilde{N[e]}(x)_p, \frac{\widetilde{e}_x(y)_q}{N^q[e](x)}\right)e(x,y)$$

which is the lattice isometry defined in Proposition 2.5 mapping $\mathbf{1}$ *to* e.

Then there exists δ *such that if* $\|\mathbf{1}-e\| < \delta$*, then for each* k*, we have that* $\|\phi(f_k) - f_k\| < \varepsilon$.

Proof. We can assume $\varepsilon < 1$. Let $K \subseteq [0,1]$ be a closed set such that for $1 \leq k \leq n+1$, $g_k|_K$ is continuous and $\mu(K) > 1-\varepsilon$. Pick $\delta' < \varepsilon$ such that for any $x, x' \in K$, if $|x - x'| < \delta'$, then $|g_k(x) - g_k(x')| < \varepsilon/4$. Now, let $\delta < {\delta'}^{2p}$ be such that $1 - \frac{\delta'}{4} \leq (1-\delta)^p < (1+\delta)^p < 1 + \frac{\delta'}{4}$, and suppose $\|\mathbf{1} - e\| < \delta$. Observe that for each x, we have $\widetilde{N[\mathbf{1}-e]}(x)_p < \delta$. For each $1 \leq k \leq n$, let

$$\widetilde{f}_n(x,y) = f\left(\widetilde{N[e]}(x)_p, \frac{\widetilde{e}_x(y)_q}{N^q[e](x)}\right).$$

Observe that $\|\widetilde{f}_k - \phi(f_k)\| < \delta < \varepsilon/4$, so it is enough to show that $\|\widetilde{f}_k - f_k\|$ is sufficiently small as well.

To this end, first note that since f is being composed with increasing continuous functions in both arguments, each $\widetilde{f}_n(x,\cdot)$ is also the characteristic function of an interval: indeed, we have piecewise continuous $\widetilde{g}_1, ..., \widetilde{g}_{n+1}$ with $\widetilde{g}_k(x) := g(\widetilde{N[e]}(x)_p)$ and $\widetilde{g}_{n+1}(x) = 1$ such that for each k, $\widetilde{f}_k(x,y) = \mathbf{1}_{[\widetilde{g}_k(x), \widetilde{g}_{k+1}(x)]}(y)$. Also observe that for $M := \{x \in K : N[e-1](x) < \delta\}$, we have $\mu(M)) > 1 - \delta' - \varepsilon$. In addition, as

$$\|f_k - \widetilde{f}_k\|^p = \|N[f_k - \widetilde{f}_k]\|_p^p = \int \mu(D(x))^p \, dx,$$

Where $D_k(x) = \{y : f_k(x,y) \neq \widetilde{f}_k(x,y)\}$. The above set up, in combination with the triangle inequality properties of N, leads us to the following inequalities:

- For all $0 \leq x \leq 1$, $|\widetilde{N[e]}(x)_p - x| < \delta$.
- For all $x \in M$, $|N[e](x) - 1| < \delta$.
- For all $x \in M$ and $0 \leq y \leq 1$, $|\widetilde{e}_x(y)_q - y| < \frac{\delta'}{2}$.
- For all $x \in M$ and $0 \leq y \leq 1$, if $y' := \frac{\widetilde{e}_x(y)_q}{N^q[e](x)}$, then $|y' - e_x(y)_q| < \frac{\delta'}{2}$ (which implies with the above that $|y - y'| < \delta'$).

We now show that the above implies that $D_k(x) < 2\varepsilon$. Observe first that for all $x \in M$, if $f_k(x,y) \neq \widetilde{f}_k(x,y)$ it must be because, but $y' \notin [\widetilde{g}_k(x), \widetilde{g}_{k+1}(x)]$, or vice versa. In either case, it can be shown that either $|y - g_k(x)| < \delta + \frac{\varepsilon}{4}$ or $|y - g_{k+1}(x)| < \delta + \frac{\varepsilon}{4}$. Suppose $y \in [g_k(x), g_{k+1}(x)]$ and $y' < \widetilde{g}_k(x)$ (a similar proof will work in the case that $y' > \widetilde{g}_{k+1}(x)$. Then since $y > g_k(x)$, $|y - y'| \leq \delta'$, and $|g_k(x) - \widetilde{g}_k(x)| < \frac{\varepsilon}{4}$,

$$0 \leq y - g_k(x) = (y - y') + (y' - \widetilde{g}_k(x)) + (\widetilde{g}_k(x) - g_k(x)) < \delta + \frac{\varepsilon}{4}.$$

It follows then that accounting for both ends of the interval $[g_k(x), g_{k+1}(x)]$ and for $x \in M$, we have $D_k(x) < 2\varepsilon$. Resultantly,

$$\|f_k - \widetilde{f}_k\|^p = \int_M \mu(D(x))^p \, dx + \int_{M^c} \mu(D(x))^p \, dx < (2\varepsilon)^p + \delta^p < 3\varepsilon^p,$$

which can be made arbitrarily small.

□

Theorem 5.2. *Let $1 \leq p \neq q < \infty$ and suppose $L_p(L_q)$ is AUH over its finite dimensional sublattices. Let $f_i : E \to L_p(L_q)$ be lattice embeddings with $E =< e_1, ..., e_n >$ such that $f_i(x) = \mathbf{1}$ for some $x \in E$. Then for all $\varepsilon > 0$, there exists an automorphism ϕ fixing $\mathbf{1}$ such that $\|\phi f_1 - f_2\| < \varepsilon$.*

Proof. Assume the above, and pick $E' =< e'_1, ..., e'_m >\subseteq L_p(L_q)$, where $e'_k = a_k \cdot \mathbf{1}_{A_k \times B_k}$ with A_k and B_k intervals such that $\sum_k \mathbf{1}_{A_k \times B_k} = \mathbf{1}$ and for each e_k there is $x_k \in S(E')_+$ such that $\|x_k - f_2(e_k)\| < \frac{\varepsilon}{4n}$.

Since $L_p(L_q)$ is AUH, there exists an automorphism ψ such that $\|\psi f_1 - f_2\| < \delta$, where δ satisfies the conditions for $\frac{\varepsilon}{4mn}$ and each of the e'_k's in E' in Lemma 5.1. Now pick the automorphism ϕ' over $L_p(L_q)$ mapping $\mathbf{1}$ to $\psi f_1(x)$ as defined in Lemma 5.1. It follows that for each e'_k, $\|\phi'(e'_k) - e'_k\| < \frac{\varepsilon}{4mn}$, so $\|\phi'(x_k) - x_k\| < \frac{\varepsilon}{4n}$. Thus for each $e_k \in E$,

$$\begin{aligned}\|\phi' f_2(e_k) - \psi f_1(e_k)\| \leq & \|\phi'(f_2(e_k) - x_k)\| + \|\phi'(x_k) - x_k\| \\ & + \|x_k - f_2(e_k)\| + \|f_2(e_k) - \psi f_1(e_k)\| < \frac{\varepsilon}{n},\end{aligned}$$

Now let $\phi = {\phi'}^{-1} \circ \psi$ to obtain the desired automorphism; then $\|\phi f_1 - f_2\| < \varepsilon$.

□

The above can be used to show that if $L_p(L_q)$ is AUH and $f_i(E)$ contains $\mathbf{1}$ for $i = 1, 2$, then we can induce almost commutativity with automorphisms fixing $\mathbf{1}$ as well. This will allow us to reduce possible automorphisms over $L_p(L_q)$ to those that in particular fix $\mathbf{1}$. The importance of this result is that these particular homomorphisms fixing $\mathbf{1}$ must always preserve base-equimeasurability for characteristic functions, as shown in Proposition 3.1. Thus a natural approach in disproving that $L_p(L_q)$ is AUH would involve finding sublattices containing $\mathbf{1}$ which are lattice isometric but whose generating elements are not base-equimeasurable. The following results do exactly that:

Lemma 5.3. *Lemma 4.2 fails when $r := p/q \in \mathbb{N}$. In particular, there exists a non-zero measure $\nu := \alpha - \beta$, with α and β positive measures such that for all polynomials P of degree $j \leq r$,*

$$\int_0^1 P(x)\ d\nu(x) = 0.$$

Remark 5.4. It is already known that a counter-example exists for $L_r(0,\infty)$ for all $r \in \mathbb{N}$, with

$$d\nu(x) = e^{-u^{\frac{1}{4}}} \sin(u^{\frac{1}{4}})\ du$$

(see [12] and [8] for more details).

.

Here we provide another example over the unit interval:

Proof. Fix such an r, and define a polynomial $g(x)$ of degree $r+1$ with $g(x) = \sum_0^{r+1} a_i x^i$ such that for all $0 \le j \le r$, $\int_0^1 x^j g(x)\ dx = 0$. This can be done by finding a non-trivial $a_0, ..., a_{r+1}$ in the null set of the $(n+1)\times(n+2)$ size matrix A with $A(i,j) = \frac{1}{i+j+1}$. Then let $d\nu(x) = g(x)\ dx$. Let $\alpha = \nu_+$ and $\beta = \nu_-$. Clearly α and β are finite positive Borel measures, but since $g \neq 0$, $\alpha \neq \beta$. □

Lemma 5.5. *Let $p/q \in \mathbb{N}$. Then there exists a two dimensional lattice $E =< e_1, e_2 >$ and lattice embeddings $f_i : E \to L_p(L_q)$ with $\mathbf{1} \in E$ such that $g_1(\mathbf{e})$ and $g_2(\mathbf{e})$ are not base-equimeasurable.*

Proof. Let $f(x)$ be a polynomial of degree at least $r+1$ as defined in Lemma 5.3 such that for all $0 \le k \le r$, $\int_0^1 t^k f(t)\ dt = 0$, and $\int_0^1 |f(x)|\ dx = 1$. Let $h_1(x) = \frac{1}{2} + f(x)_+$, and let $h_2(x) = \frac{1}{2} + f(x)_-$. Note that each $h_i(x) > 0$, and furthermore that $\int_0^1 h_i(t)\ dt = 1$. Additionally, each map $H_i(x) = \int_0^x h_i(t)\ dt$ is strictly increasing with $H_i(0) = 0$ and $H_i(1) = 1$. Now we will construct characteristic functions $f_j^i \in L_p(L_q)$ such that the linear map $f_j^1 \mapsto f_j^2$ induces an isometry, but $N\mathbf{f}^1$ and $\mathbf{f}^2$ are not base-equimeasurable. From there, we let $e_j = \frac{f_j^1}{\|f_j^i\|}$, and let g_i be the lattice isometry induced by $g_i(e_j) = \frac{f_j^i}{\|f_j^i\|}$,

To this end, let

$$F_1^i(x) := H_i^{-1}(x), \text{ and } F_2^i(x) := 1 - F_1^i(x).$$

Observe that $F_1^1(x) \neq F_1^2(x)$. Indeed, one can show that the associated push forwards $dF_{1\#}^i\mu$ for each F_1^i have the corresponding equivalence:

$$dF_{1\#}^i\mu(x) = h_i(x)\ dx$$

So (F_1^1, F_2^1) and (F_1^2, F_2^2) are not equimeasurable. However, For $0 \le j \le r$, $u^j h_i(u)\ du = u^j\ dF_{1\#}^i(u) = F_1^i(x)^j\ dx$, so it follows from the construction of the h_i's that

$$\int_0^1 F_1^1(x)^j\ dx = \int_0^1 F_1^2(x)^j\ dx.$$

Thus for any $v_1, v_2 > 0$, since F_1^i and F_2^i are both positive, we have

$$\int_0^1 |v_1 F_1^1(x) + v_2 F_2^1(x)|^r \, dx = \int_0^1 ((v_1 - v_2)F_1^1(x) + v_2)^r \, dx$$
$$= \sum_0^r \binom{r}{j} (v_1 - v_2)^j v_2^{r-j} \int_0^1 F_1^1(x)^j \, dx = \int_0^1 |v_1 F_1^2(x) + v_2 F_2^2(x)|^r \, dx$$

To conclude the proof, let $f_1^i(x,y) = \mathbf{1}_{[0,F_1^i(x)]}(y)$, and let $f_2^i = \mathbf{1} - f_1^i$. Clearly $N[f_j^i] = F_j^i$. □

Theorem 5.6. *If $p/q \in \mathbb{N}$ and $p \neq q$, then $L_p(L_q)$ is not AUH for the class of its finite dimensional sublattices.*

Proof. Fix $p/q \in \mathbb{N}$, and let E be the 2-dimensional lattice generated in Lemma 5.5, with $f_i : E \to L_p(L_q)$ embeddings mapping to copies of $E =< e_1, e_2 >$ such that $f_1(\mathbf{e})$ and $f_2(\mathbf{e})$ are not base-equimeasurable. In addition, by assumption $\mathbf{1} \in E$. For notational ease, let $F_j^i = N[f_i(e_j)]$.

Suppose for the sake of contradiction that $L_p(L_q)$ is AUH. Pick some measurable $C \subseteq [0,1]^2$ and $\varepsilon > 0$ such that

$$* \quad \mathbf{F}^2_{\#}\mu(C) > \mathbf{F}^1_{\#}\mu(C+\varepsilon) + \varepsilon,$$

where

$$C + \varepsilon = \{\mathbf{t} \in [0,1]^2 : \|\mathbf{t} - \mathbf{s}\|_\infty < \varepsilon \text{ for some } \mathbf{s} \in C\}.$$

By Theorem 5.2, there is some lattice automorphism $\phi : L_p(L_q) \to L_p(L_q)$ fixing $\mathbf{1}$ such that $\|\phi \circ f_1 - f_2\| < \varepsilon^2$. Let $\phi F_j^i = N[\phi f_i(e_j)]$. By Proposition 3.1, ϕ preserves base-equimeasurability, so for any measurable B,

$$\phi \mathbf{F}^1_{\#}\mu(B) = \mathbf{F}^1_{\#}\mu(B).$$

By the properties of N, we also have $\|\phi F_j^1 - F_j^2\|_p \leq \|\phi f_1(e_j) - f_2(e_j)\|$. It also follows that

$$\mu(t : \|\phi \mathbf{F}^1(t) - \mathbf{F}^2(t)\|_\infty > \varepsilon) < \varepsilon,$$

so $\phi \mathbf{F}^1_{\#}\mu(C+\varepsilon) + \varepsilon > \mathbf{F}^2_{\#}\mu(C)$, but this contradicts the assumption (*). So Theorem 5.2 cannot apply, implying that $L_p(L_q)$ is not AUH as desired. □

Remark 5.7. For $p/q \in \mathbb{N}$, $L_p(L_q)$ is the unique lattice that is separably AUH over finitely generated BL_pL_q spaces, since up to isometry it is the unique doubly atomless BL_pL_q space. In light of Theorem 5.6, this implies that the class of finitely generated sublattices of $L_p(L_q)$ is not a Fraïssé class as defined in [2], as $L_p(L_q)$ is the only possible candidate as a Fraïssé limit.

In particular, L_pL_q lacks the NAP. Indeed, otherwise, one can use that NAP with BL_pL_q amalgamate lattices and [7, Proposition 2.8] to situate a $d^{\mathcal{K}}$-Cauchy sequence into a Cauchy-sequence of generating elements in an ambient separable BL_pL_q lattice. Thus $\overline{\mathcal{K}_{p,q}}$ would also have the Polish

Property, implying that $\overline{\mathcal{K}_{p,q}}$ is a Fraïssé class. Since the only possible candidate Fraïssé limit space is $L_p(L_q)$ itself, this would contradict Theorem 5.6.

References

[1] S. Banach. Théorie des opérations linéaires. 1932.

[2] I. Ben Yaacov. Fraïssé limits of metric structures. *J. Symb. Log.*, 80(1):100–115, 2015.

[3] M. Cambern. The isometries of $L_p(X, K)$. *Pacific Journal of Mathematics*, 55(1):9–17, 1974.

[4] M. de Jeu. Determinate multidimensional measures, the extended Carleman theorem and quasi-analytic weights. *Annals of Probability*, 31:1205–1227, 2001.

[5] V. Ferenczi, J. Lopez-Abad, B. Mbombo, and S. Todorcevic. Amalgamation and Ramsey properties of L_p spaces. *Adv. Math.*, 369:107190, 76, 2020.

[6] C. W. Henson and Y. Raynaud. On the Theory of $L_p(L_q)$-Banach lattices. *Positivity*, 11(2):201–230, 2007.

[7] C. W. Henson and Y. Raynaud. Quantifier elimination in the theory of $L_p(L_q)$-Banach lattices. *J. Log. Anal.*, 3:Paper 11, 29, 2011.

[8] W. Linde. Uniqueness theorems for measures in L_r and $C_0(\omega)$. *Mathematische Annalen*, 274:617–626, 1986.

[9] M. Lupini. Fraïssé limits in functional analysis. *Adv. Math.*, 338:93–174, 2018.

[10] W. Lusky. Some consequences of Rudin's paper" L_p-isometries and equimeasurability". *Indiana University Mathematics Journal*, 27(5):859–866, 1978.

[11] Y. Raynaud. Sur les sous-espaces de $L_p(L_q)$, Séminaire d'Analyse fonctionnelle paris vii–vi, année 1984–85. *Publ. Math. Univ. Paris VII*, 26:49–71, 1986.

[12] W. Rudin. L_p-isometries and equimeasurability. *Indiana University Mathematics Journal*, 25(3):215–228, 1976.

[13] A. R. Sourour. On the isometries of $L^p(\{\Omega, X\})$. *Bulletin of the American Mathematical Society*, 83:129–130, 1977.

[14] M. A. Tursi. A separable universal homogeneous Banach lattice. *International Mathematics Research Notices*, 2023(7):5438–5472, 2023.